Exam Strongly Correlated Quantum Systems

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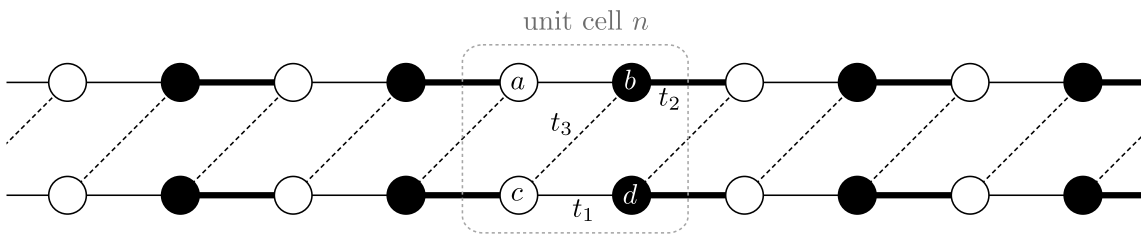
1 One-dimensional topological insulators

Consider an infinitely long, one-dimensional lattice with four fermionic modes per unit cell, with the following Hamiltonian:

$$\begin{aligned}\hat{H} = \sum_{n\in\mathbb{Z}} \Big[&- t_1(a_n^\dagger b_n + b_n^\dagger a_n + c_n^\dagger d_n + d_n^\dagger c_n) \\ &- t_2(a_{n+1}^\dagger b_n + b_n^\dagger a_{n+1} + c_{n+1}^\dagger d_n + d_n^\dagger c_{n+1}) \\ &- t_3(c_n^\dagger b_n + b_n^\dagger c_n + d_n^\dagger a_{n+1} + a_{n+1}^\dagger d_n)\Big]\end{aligned} \tag{1}$$

where the parameters t_1, t_2 and t_3 can be assumed real and nonnegative. The following figure depicts the geometry of this model.



1. Consider the three extreme cases where only one of the three parameters t_1, t_2 or t_3 is nonzero. Can you schematically describe the structure of the ground state in those three cases?

2. Now transform to momentum space via

$$a_n = \int_{-\pi}^{+\pi} A(k)\mathrm{e}^{+ikn}\,\mathrm{d}k \tag{2}$$

and similarly for b_n, c_n, d_n and their adjoints, and write the Hamiltonian as

$$\hat{H} = \int_{-\pi}^{+\pi} \begin{bmatrix} A(k)^\dagger & B(k)^\dagger & C(k)^\dagger & D(k)^\dagger \end{bmatrix} \mathsf{h}(k) \begin{bmatrix} A(k) \\ B(k) \\ C(k) \\ D(k) \end{bmatrix} \mathrm{d}k \tag{3}$$

using some 4×4 matrix $\mathsf{h}(k)$.

3. This Hamiltonian possesses a chiral or sublattice symmetry U_C such that $\mathsf{U}_C \mathsf{h}(k) \mathsf{U}_C^\dagger = -\mathsf{h}(k)$, with $\mathsf{U}_C = \mathsf{U}_C^\dagger = \mathsf{U}_C^{-1}$. Can you find a choice of U_C? By suitably reordering the modes, you should be able to bring U_C into the standard form $\mathsf{U}_C = \mathrm{diag}(\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix})$ and
$$\mathsf{h}(k) = \begin{bmatrix} 0 & \mathsf{q}(k)^\dagger \\ \mathsf{q}(k) & 0 \end{bmatrix}. \tag{4}$$

4. It is *not* straightforward to analytically compute the spectrum of $\mathsf{h}(k)$ for arbitrary values of the parameters. The eigenvalues of $\mathsf{h}(k)$ correspond to the singular values of $\mathsf{q}(k)$. Hence, $\mathsf{h}(k)$ will have eigenvalues zero (for some k) if $\det(\mathsf{q}(k)) = 0$ (for some k). When $\det(\mathsf{q}(k)) \neq 0$ for all $k \in [-\pi, +\pi)$, the complex phase of $\det(\mathsf{q}(k))$ determines a winding number that is a topological invariant of this system. Setting $\alpha = t_1/t_2$ and $\beta = t_3/t_2$, determine the regions in $(\alpha, \beta) \in [0, +\infty)^{\times 2}$ (the upper right quadrant) where $\det(\mathsf{q}(k))$ can become zero for some k. The complement thereof are the regions where $\hat{H}$ is gapped (in the bulk) and the winding number is well defined. Determine this winding number. Can you relate its value to edge modes that would exist in a finite system, for example in the extreme limits from question 1?
Hint: make some plots of $\det(\mathsf{q}(k))$ as parametric curve in the complex plane for different values of β and for $\alpha = 0$, $\alpha = 1/2$, $\alpha = 1$ and $\alpha = 2$ to get some intuition and to see the winding number emerge.

5. Consider now the alternative Hamiltonian
$$\begin{aligned}\hat{H} = \sum_{n\in\mathbb{Z}} \Big[&- t_1 (a_n^\dagger b_n + b_n^\dagger a_n + c_n^\dagger d_n + d_n^\dagger c_n) \\ &- t_2 (a_{n+1}^\dagger b_n + b_n^\dagger a_{n+1} + c_{n+1}^\dagger d_n + d_n^\dagger c_{n+1}) \\ &- t_4 (c_n^\dagger a_n + a_n^\dagger c_n + d_n^\dagger b_n + b_n^\dagger d_n)\Big]\end{aligned} \tag{5}$$
For the value $t_1 = 0$, is it possible to connect the ground state at $t_4/t_2 = 0$ to that at $t_4/t_2 = +\infty$ without crossing a phase transition? What happens to the edge modes in the case of a finite system?

2 SYMMETRY PROTECTED TOPOLOGICAL ORDER AND DUALITIES

Consider an infinitely long, one-dimensional lattice of qubits or spin-1/2 particles, with a Hamiltonian given by
$$\hat{H}(g) = -\sum_{n\in\mathbb{Z}} \sigma_{n-1}^x \sigma_n^z \sigma_{n+1}^x - g \sum_{n\in\mathbb{Z}} \sigma_n^z \tag{6}$$

1. We first consider the Hamiltonian with $g = 0$, i.e. only the first type of terms. Note that these individual terms are mutually commuting. Nonetheless, the ground state is not entirely trivial. To construct the ground state, we will first block every two sites together into a single unit cell, i.e. unit cell n consists of sites $2n$ and $2n + 1$. It is then useful to explicitly write the terms in the Hamiltonian as
$$\hat{H}(0) = -J_1 \sum_{n\in\mathbb{Z}} (1_{2n-2} \otimes \sigma_{2n-1}^x) \otimes (\sigma_{2n}^z \otimes \sigma_{2n+1}^x) + (\sigma_{2n}^x \otimes \sigma_{2n+1}^z) \otimes (\sigma_{2n+2}^x \otimes 1_{2n+3}). \tag{7}$$

Now consider the unitary transformation U of two qubits given by the matrix

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \tag{8}$$

Show that the following equalities hold:

$$u^\dagger(\sigma^z \otimes \sigma^x)u = (\sigma^x \otimes 1), \qquad u^\dagger(\sigma^x \otimes \sigma^z)u = (1 \otimes \sigma^z), \tag{9}$$

$$u^\dagger(\sigma^x \otimes 1)u = (\sigma^z \otimes 1), \qquad u^\dagger(1 \otimes \sigma^x)u = (1 \otimes \sigma^x). \tag{10}$$

2. If we now apply u from Eq. (8) to every unit cell and thus set $U = \bigotimes_{n\in\mathbb{Z}} u_n$, it follows directly from the identities in Eq. (8) that the Hamiltonian in Eq. (7) transforms into

$$U^\dagger \hat{H}(0) U = -J_1 \sum_{n\in\mathbb{Z}} (1_{2n-2} \otimes \sigma^x_{2n-1}) \otimes (\sigma^x_{2n} \otimes 1_{2n+1}) + (1_{2n} \otimes \sigma^z_{2n+1}) \otimes (\sigma^z_{2n+2} \otimes 1_{2n+3}). \tag{11}$$

Observing that $|00\rangle + |11\rangle$ is the unique ground state of $-(\sigma^x \otimes \sigma^x + \sigma^z \otimes \sigma^z)$, it should now be possible to combine these observations to write the ground state of the Hamiltonian $U^\dagger \hat{H} U$, and thus also of $\hat{H}$, as a matrix product state. Every MPS tensor is associated to a unit cell, and thus there are 4 matrices A^{00}, A^{10}, A^{01} and A^{11} that need to be determined.

3. Still considering $\hat{H}$ on the two-site unit cell, we can observe that it has two independent (commuting) $\mathbb{Z}_2$ symmetries, namely

$$Z_1 = \bigotimes_{n\in\mathbb{Z}} (\sigma^z_{2n} \otimes 1_{2n+1}), \qquad Z_2 = \bigotimes_{n\in\mathbb{Z}} (1_{2n} \otimes \sigma^z_{2n+1}). \tag{12}$$

The ground state is unique and thus symmetric under this $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group. Find the gauge transforms associated with those symmetries acting on the virtual level of the MPS. Is this a linear or a (nontrivial) projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$?

4. Now consider the Hamiltonian with nonzero values of g. Does it still has Z_1 and Z_2 as symmetry? What is the ground state in the limit $g \to \infty$? Do you expect a phase transition for some value of g? Why?

5. For this question, it is easier to temporarily conjugate the Hamiltonian with $V = \bigotimes_{n\in\mathbb{Z}} v_n$ with $v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ the Hadamard gate, which has the effect of interchanging $\sigma^x \leftrightarrow \sigma^z$, i.e. $v = v^\dagger = v^{-1}$ and $v\sigma^x v = \sigma^z$. We thus have

$$\hat{H}'(g) = V\hat{H}(g)V^\dagger = -\sum_{n\in\mathbb{Z}} \sigma^z_{n-1}\sigma^x_n\sigma^z_{n+1} - g\sum_{n\in\mathbb{Z}} \sigma^x_n \tag{13}$$

Now consider the two-site unitary transformation

$$w_{n,n+1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{14}$$

Because w is diagonal, it commutes with any product of σ^z operators, as well as with other w transformations acting on different sites, i.e. $[w_{n,n+1}, w_{m,m+1}] = 0$ for all $n, m \in \mathbb{Z}$. As such, the product $W = \prod_n w_{n,n+1}$ of all w operators is well defined and independent on a chosen order. Show that conjugation with W has the effect of interchanging the two types of terms in $\hat{H}'(g)$ from Eq. (13), i.e. $W^\dagger \hat{H}'(g) W = g\hat{H}'(1/g)$. What does this tell you about the position of a possible phase transition?

6. The Hamiltonian $\hat{H}$ can be diagonalised exactly using the Jordan-Wigner, Fourier and Bogoliubov transforms from the lecture notes. Use these to compute the dispersion relation of this Hamiltonian, and to verify that indeed the gap closes at the value found in the previous question.

7. Now consider a different mapping of the Pauli operators, which is given by

$$\sigma^z_{2n} = \tau^z_{2n}, \qquad \sigma^x_{2n} = \prod_{m \leq n} \tau^z_{2m-1} \tau^x_{2n}, \tag{15}$$

$$\sigma^z_{2n+1} = \tau^z_{2n+1}, \qquad \sigma^x_{2n+1} = \tau^x_{2n+1} \prod_{m > n} \tau^z_{2m}. \tag{16}$$

Show that this mapping respects the Pauli algebra $[\sigma^x_m, \sigma^z_n] = 0$ for all $m \neq n$, $[\sigma^z_m, \sigma^z_n] = 0$ and $[\sigma^x_m, \sigma^x_n] = 0$ for all $m, n \in \mathbb{Z}$, and $\{\sigma^x_n, \sigma^z_n\} = 0$ for all $n \in \mathbb{Z}$. Show that, by inserting this transformation in $\hat{H}(g)$, the Hamiltonian in terms of the τ variables becomes that of two decoupled quantum Ising models with transverse magnetic field. Is this compatible with the position and nature of the phase transition you have found? What happens with the ground state degeneracy?