Groups & Representations
Exercise session 1

February 25, 2022
14:30-17:30

1 Symmetries in classical and quantum systems

Symmetries in physics

Symmetries form the cornerstone of our temporary understanding of physics. A symmetry is an action on a physical system that leaves the dynamics, ie. the equations of motion, invariant. Symmetries appear all over the place in physics, most notably in quantum physics and relativity theory, but also in classical physics symmetries are often a tool to gain insight in the dynamics of the system and sometimes even provide a way to solve the system completely.

1.1. Consider a Hamiltonian system described by the (independent) coordinates $\{q^i, p_i | i = 1, 2, ..., N\}$ and a generic Hamiltonian $H \equiv H(q^i, p_i)$. Recall that the Hamilton equations governing the dynamics of such a system read

$$\begin{cases} \dot{q}^i &= \frac{\partial H}{\partial p_i} \qquad i = 1, 2, ..., N \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} \qquad i = 1, 2, ..., N \end{cases}. \tag{1}$$

(a) Define $\vec{\eta} := (q^1, ..., q^N, p_1, ..., p_N)$ to write the Hamilton equations as

$$\dot{\eta}^i = \sum_j \Omega_{ij} \frac{\partial H}{\partial \eta^j}, \tag{2}$$

where $\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$.

(b) Consider a coordinate transformation of the form

$$\begin{cases} q^i &\mapsto Q^i\left(q^i, p_i\right) \\ p_i &\mapsto P_i\left(q^i, p_i\right) \end{cases}. \tag{3}$$

Which coordinates transformations preserve the form of the Hamilton equations? Such coordinate transformations are called *canonical coordinate transformations*.

1.2. Consider a Hamiltonian system as in the previous exercise. The *Poisson bracket* on differentiable functions $f\left(q^i, p_i\right)$ is defined as

$$\{f, g\} := \sum_i \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \left(p_i \leftrightarrow q^i\right). \tag{4}$$

(a) Compute:

i. $\{q^i, q^j\}$,

ii. $\{p_i, p_j\}$,

iii. $\{q^i, p_j\}$.

How does this relate to the canonical commutation relations encountered in quantum mechanics?

(b) Show that the Poisson bracket is:

i. skew-symmetric,

ii. bilinear,

and satisfies the

iii. Jacobi identity

$$\{f, \{g, h\}\} + (\text{cyclic}) = 0. \tag{5}$$

The space of functions $f(q^i, p_i)$ on phase space can be endowed with the structure of a vector space. A vector space that comes with a bilinear map that satisfies the Jacobi identity and is skew-symmetric, is called a *Lie algebra*, and plays a very important role in - among many other things - the classification of continuous groups, infinitesimal symmetry transformations in field theory,... Different Lie algebras will be encountered many times.

(c) As a concrete example, consider a three-dimensional system described by Cartesian coordinates x^1, x^2, x^3. The angular momentum (pseudo-)vector $\vec{L}$ is defined as $\vec{L} := \vec{r} \times \vec{p}$. Compute $\{L_i, L_j\}$.

(d) Show that the Poisson bracket is preserved under canonical transformations. Hint: write $\{f, g\} = \sum_{i,j} \frac{\partial f}{\partial \eta^i} \Omega_{ij} \frac{\partial g}{\partial \eta^j}$.

1.3. Consider a quantum mechanical system that evolves in time according to some time-independent Hamiltonian $\widehat{H}$, show that the expectation value of every time-independent operator that commutes with $\widehat{H}$ is conserved in time.

2 Basic properties of groups

Basic notions of group theory

A *group* G is a (possibly infinite) set $\{g_1, g_2, ...\}$ endowed with a group multiplication $\mathsf{G} \times \mathsf{G} \mapsto \mathsf{G} : (g_1, g_2) \mapsto g_1 \cdot g_2 =: g_{12}$ that satisfies following properties.

- There is a unit element for the multiplication, usually denoted by e or 1 (or 0 in case of abelian groups (see below)).
- The multiplication is associative.
- For every element $g \in \mathsf{G}$ there exists an inverse element g^{-1} such that $gg^{-1} = g^{-1}g = e$.

If the group multiplication is commutative, we call the group a *commutative* or *abelian* group. In that case the group multiplication is often denoted by $+$.

If the group is finite, the number of group elements is called the *cardinality* or *order* of the group. The *order of a group element* $g \in \mathsf{G}$ is the smallest natural number n such that $g^n = e$. We will prove below that $n \leq |\mathsf{G}|$.

The *direct product* of two groups G_1 and G_2, denoted by $\mathsf{G}_1 \times \mathsf{G}_2$, is the Cartesian product $\left\{(g_i^{(1)}, g_j^{(2)}) | g_i^{(1)} \in \mathsf{G}_1, g_i^{(2)} \in \mathsf{G}_2, i, j = 1, 2, ..., |\mathsf{G}|\right\}$ endowed with the pairwise group multiplication $\left(g_i^{(1)}, g_j^{(2)}\right) \cdot \left(g_{\tilde{i}}^{(1)}, g_{\tilde{j}}^{(2)}\right) := \left(g_i^{(1)} g_{\tilde{i}}^{(1)}, g_j^{(2)} g_{\tilde{j}}^{(2)}\right)$.

A *subgroup* H of G, $\mathsf{H} \leq \mathsf{G}$, is a subset of G that in itself forms a group.

A *normal subgroup* N of a group G, $\mathsf{N} \trianglelefteq \mathsf{G}$, is a subgroup of G such that $gN = Ng, \forall g \in \mathsf{G}$.

The *centre* or *centralizer* of a group G, $Z(\mathsf{G})$, is the normal subgroup of elements of G that commute with all the other elements of G.

A *group homomorphism* φ is a map between groups $\varphi : \mathsf{G}_1 \mapsto \mathsf{G}_2$ that is compatible with the group structure, ie. $\varphi(g_1)\varphi(g_2) = \varphi(g_{12})$. A bijective group homomorphism is called a *group isomorphism*. A *group automorphism* is a group isomorphism from a group to itself.

2.1. Which of the following examples are (possibly infinite) groups? What is the order of the groups you find? Are they abelian or not?

(a) The integers $\{0, 1, ..., N-1\}$ with group multiplication addition modulo N.
(b) The set of natural numbers $\mathbb{N}$ with group multiplication addition.
(c) The integers $\mathbb{Z}$ with group multiplication addition.
(d) The permutations of a set of N elements with group multiplication the composition of functions.
(e) $\mathsf{SO}_n := \left\{M \in \mathbb{R}^{n \times n} | \det M = 1, M^{-1} = M^\top\right\}$ with group multiplication the ordinary matrix multiplication.

Groups & Representations
Exercise session 2

March 11, 2022
14:30-17:30

1 Basic properties of groups

1.1. Construct D_4. What is the order of this group? Write down its generators and its Cayley table. Hint: recall from the lectures that D_4 is the symmetry group of a regular 4-gon, aka. a square.

1.2. *Rearrangement theorem:* Given some finite group G, show that every row and column of the Cayley table contains every group element exactly once.

1.3. (a) Given some finite group G, prove that for every $g \in \mathsf{G}$ the order of g is smaller than or equal to the order of the group. Hint: look at the set $\{1, g, g^2, ...\}$.

(b) Argue that every finite group has a cyclic subgroup.

(c) Argue that the order of every group element is a divisor of the order of the group. Hint: use Lagrange's theorem.

1.4. Show there are two groups of order four (up to isomorphism). Hint: use (1.3c).

2 Automorphisms

Group homomorphisms and the automorphism group

A *group homomorphism* φ is a map between groups $\varphi : \mathsf{G}_1 \mapsto \mathsf{G}_2$ that is compatible with the group structure, ie. $\varphi(g_1)\varphi(g_2) = \varphi(g_{12})$. A bijective group homomorphism is called a *group isomorphism.* A *group automorphism* is a group isomorphism from a group to itself.

2.5. Compute following automorphism groups

(a) $\mathsf{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$

(b) $\mathsf{Aut}(\mathbb{Z}_8)$

Groups & Representations
Exercise session 3

April 1, 2022
16:00-17:30

1 Center subgroup, classes, quotient group

Center, classes, cosets, normal subgroups and quotient groups

The *center* of a group G, $Z(\mathsf{G})$, is the subgroup consisting of group elements that commute with all the elements of the group, $Z(\mathsf{G}) := \{h \in \mathsf{G} | hg = gh, \forall g \in \mathsf{G}\}$.

The *conjugacy class* associated to a group element $g \in \mathsf{G}$, $c(g)$, is given by

$$c(g) := \left\{hgh^{-1} | h \in \mathsf{G}\right\}. \tag{1}$$

Given some group G and a subgroup $\mathsf{H} \leq \mathsf{G}$ one can define the

- *left cosets* $g\mathsf{H} = \{gh : h \in \mathsf{H}\}$ for all $g \in \mathsf{G}$,
- *right cosets* $\mathsf{H}g = \{hg : h \in \mathsf{H}\}$ for all $g \in \mathsf{G}$.

Given some coset $g\mathsf{H}, \mathsf{H}g$, g is called a *representative* of the coset. Note that there are always $g_1 \neq g_2$ such that $g_1\mathsf{H} = g_2\mathsf{H}$.
A *normal subgroup*, $\mathsf{N} \trianglelefteq \mathsf{G}$, is a subgroup $\mathsf{N} \leq \mathsf{G}$ such that $gng^{-1} \in \mathsf{N}, \forall n \in \mathsf{N}, \forall g \in \mathsf{G}$. The center of a group is normal (see extra exercises).

In case we have a normal subgroup $\mathsf{N} \trianglelefteq \mathsf{G}$, the left - and right cosets coincide, $g\mathsf{N} = \mathsf{N}g$ for all $g \in \mathsf{G}$ (think about this!).

Given a group G and a normal subgroup thereof, $\mathsf{N} \trianglelefteq \mathsf{G}$, the *quotient* - or *factor group* G/N is the group of left (or right) cosets $g\mathsf{N}$, together with the group multiplication $g_1\mathsf{N} \cdot g_2\mathsf{N} := (g_1g_2)\mathsf{N}$. This group multiplication is well defined in the sense that it does not depend on the choice of representatives for the cosets (see extra exercises).

1.1. The quaternion group Q is a group of order 8. Its group elements are $\{\pm 1, \pm i, \pm j, \pm k\}$ and the multiplication is given by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. It is clearly a non-abelian group as eg. $ij \neq ji$.[1]

   (a) Compute the center $Z(\mathsf{Q})$. Verify it is a subgroup. Which one?
   (b) Construct the conjugacy classes of Q.
   (c) Argue that the order of all the elements in a given class is the same. Prove this in general.

[1] Give an argument why $\mathsf{Q} \ncong \mathsf{D}_4$.

(d) Show that $\{1,-1\} \triangleleft \mathsf{Q}$ and construct $\mathsf{Q}/\{1,-1\}$.

(e) Same exercise for $\{\pm 1, \pm i\} \triangleleft \mathsf{Q}$.

2 Semidirect product

Semidirect products

Given two groups N and H, the *semidirect product* of N and H with respect to β, written as

$$\mathsf{G} = \mathsf{N} \rtimes_\beta \mathsf{H}, \tag{2}$$

is the group with group elements $\mathsf{N} \times \mathsf{H}$, ie. all the elements of G are of the form (n,h) for $n \in \mathsf{N}$ and $h \in \mathsf{H}$, and with group multiplication defined as

$$(n_1,h_1)\cdot(n_2,h_2) := (n_1\beta_{h_1}(n_2), h_1h_2). \tag{3}$$

Here β is a homomorphism $\beta : \mathsf{H} \mapsto \mathsf{Aut}(N)$.
$\mathsf{N} = \{(n, e_H)\}$ then constitutes a normal subgroup of G whereas $\mathsf{H} = \{(1_N, h)\}$ forms a (not necessarily normal) subgroup of G.

2.2. (a) Construct $\mathsf{Aut}(\mathbb{Z}_3)$.

(b) Construct all homomorphisms $\beta : \mathbb{Z}_2 \mapsto \mathsf{Aut}(\mathbb{Z}_3)$.

(c) For all the β's you found, construct $\mathbb{Z}_3 \rtimes_\beta \mathbb{Z}_2$.

3 Represention**s**

Representations

A *representation* of a finite group G is a group homomorphism $\rho : \mathsf{G} \mapsto \mathsf{GL}(V)$, V being a (finite dimensional) vector space. I.e. ρ maps every group element $g \in \mathsf{G}$ to some matrix X_g such that

$$X_{g_1}X_{g_2} = X_{g_{12}}, \qquad \forall g_1, g_2 \in \mathsf{G}. \tag{4}$$

$\dim(V)$ is the *dimension of the representation.*

Notice that $X_e = \mathbb{1}$.
A representation is *faithful* if ρ is injective.

3.3. Show that the Pauli matrices

$$\sigma_X = \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_Y = \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_Z = \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

together with the 2×2 identity matrix can be used to construct a faithful two-dimensional representation of Q.

Groups & Representations
Exercise session 4

April 22, 2022
16:00-17:30

1 Basic notions of the representation theory of finite groups

1.1. (a) Show that all the irreducible representations of an abelian group are one-dimensional.

(b) Construct all irreps of $\mathbb{Z}_k$. Check that there are as many irreps as there are conjugacy classes. Of particular importance are the irreps of $\mathbb{Z}_2$, which are these?

1.2. Consider the group $S_3 \cong \mathsf{D}_3$. In the lecture notes it is explained how to derive the character table from some general arguments and constraints. In this exercise we will construct the irreps that realize these characters. First we recall the *cycle notation* of permuations that was introduced in the lecture notes towards the end of the chapter:

The cycle notation of the permutation group

A *cycle* is a cyclic map of a some 'letters' into each other: eg. consider the following permutation of 5 elements (i.e. element of S_5):

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}. \tag{1}$$

This permutation maps the elements in the top row to the corresponding elements in the bottom row. It has one cycle of length 3 (3-cycle) denoted by (245), the meaning of this is that 2 is mapped to 4, 4 to 5 and 5 back to 2. (452) and (524) also denote this same cycle. It also has a 2-cycle (13) = (31).
Every permutation can be written as a product of disjoint cycles, meaning the cycles don't have elements in common. In the example above: $\sigma = (13)(245)$
All permutations that have the same *cycle stucture*, meaning they have the same number of 1-cycles, 2-cycles,... belong to the same conjugacy class (convince yourself of that).

(a) Write down all elements of S_3 as cycles.

(b) How many classes does S_3 have? How many irreps do you expect?

(c) Construct the *defining* representation of S_3. This representation is defined as $[D(p)]_{ij} = 1$ if $i = p(j)$ and 0 otherwise, $D(p) = \sum_j |p(j)\rangle\langle j|$. You do not have to check that this indeed forms a representation, but this can be a useful exercise to check.

(d) Consider following basis transformation:

$$\tilde{\vec{e}}_1 = \frac{1}{\sqrt{3}}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \tag{2a}$$

$$\tilde{\vec{e}}_2 = \frac{1}{\sqrt{6}}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) \tag{2b}$$

$$\tilde{\vec{e}}_3 = \frac{1}{\sqrt{2}}(\vec{e}_2 - \vec{e}_3). \tag{2c}$$

Write down the basis transformation in matrix notation: $\tilde{\vec{e}}_i = \sum_j C_{ij}\vec{e}_j$. Write down the inverse matrix C^{-1}.

(e) Apply the basis transformation $CR(g)C^{-1}$ to the defining representation. Which irreps do you find?

(f) Compute the characters of the irreps you found in the previous exercise. Make a character table of S_3.

(g) Construct the last missing irrep from the character table.

(h) Give a geometrical interpretation of the two-dimensional irrep.

(i) Which of these irreps are faithful?

2 Tensor product of representations

Tensor products

Given a d-dimensional vector space V with orthogonal basis $\{|\psi_i\rangle \,|i = 1, 2..., d\}$, the tensor product $V \otimes V$ is a d^2-dimensional vector space with orthogonal basis $\{|\psi_i\rangle \otimes |\psi_j\rangle \,|i, j = 1, 2..., d\}$. We will simply write $|\psi_i\rangle \otimes |\psi_j\rangle \equiv |\psi_i\rangle |\psi_j\rangle \equiv |\psi_{ij}\rangle$.

Given linear operators A and B acting on V, $A \otimes B$ acts on $V \otimes V$ according to $(A \otimes B) |\psi_{ij}\rangle = A |\psi_i\rangle \otimes B |\psi_j\rangle$.

$V \otimes V$ contains two particularly interesting linear subspaces:

(a) The symmetrized subspace $(V \otimes V)^S$ with orthogonal basis vectors $\left|\psi_{ij}^S\right\rangle :=$ $|\psi_{ij}\rangle + |\psi_{ji}\rangle$, where $i, j = 1, 2, ..., d$, which is symmetric under $i \leftrightarrow j$: $\left|\psi_{ij}^S\right\rangle =$ $\left|\psi_{ji}^S\right\rangle$.

(b) The antisymmetrized subspace $(V \otimes V)^A$ with orthogonal basis vectors $\left|\psi_{ij}^A\right\rangle := |\psi_{ij}\rangle - |\psi_{ji}\rangle$, where $i, j = 1, 2, ..., d$, which is skewsymmetric under $i \leftrightarrow j$: $\left|\psi_{ij}^A\right\rangle = -\left|\psi_{ji}^A\right\rangle$.

$V \otimes V = (V \otimes V)^S \oplus (V \otimes V)^A$.

If we are given some matrix representation of A and B, $\mathbf{A}$ and $\mathbf{B}$, with respect to the basis vectors $|\psi_i\rangle$, then the matrix representation of $A \otimes B$ with respect to the basis $\{|\psi_i\rangle \otimes |\psi_j\rangle \,|i, j = 1, 2..., d\}$ is given by the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \otimes \mathbf{B}$.

An example from physics

Fundamental particles that appear in nature are either fermions (particles with half integer spin $\frac{1}{2}$,$\frac{3}{2}$,..., the matter particles) or bosons (particles with integer spin 0,1,2,..., the force carriers such as the photon). See further courses on particle physics, statistical physics, QFT,...

Suppose we were given two identical non-interacting bosons described by wavefunctions $\left|\psi^{(1)}\right\rangle$ and $\left|\psi^{(2)}\right\rangle$ respectively, the wavefunction of the combined system boson 1 + 2 is given by $\left|\psi^{(12)}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\psi^{(1)}\right\rangle \left|\psi^{(2)}\right\rangle + \left|\psi^{(2)}\right\rangle \left|\psi^{(1)}\right\rangle\right)$ and thus lives in the symmetric subspace.
If, on the other hand, we were given two identical non-interacting fermions described by wavefunctions $\left|\psi^{(1)}\right\rangle$ and $\left|\psi^{(2)}\right\rangle$ respectively, the wavefunction of the combined system fermion 1 +2 is given by $\left|\psi^{(12)}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\psi^{(1)}\right\rangle \left|\psi^{(2)}\right\rangle - \left|\psi^{(2)}\right\rangle \left|\psi^{(1)}\right\rangle\right)$ and thus lives in the antisymmetric subspace.

2.3. (a) Given a representation $\{U(g)|g \in \mathsf{G}\}$ of a (finite) group, show that the tensor product representation $\{U(g) \otimes U(g)|g \in \mathsf{G}\}$ also forms a representation of this group. What is its dimension?

(b) What are the characters of the tensor product representation in function of the characters of the representation $\{U(g)|g \in \mathsf{G}\}$?

(c) Determine the dimension of $(V \otimes V)^S$ and $(V \otimes V)^A$.

(d) Show that $(V \otimes V)^S$ and $(V \otimes V)^A$ constitute subrepresentations of $V \otimes V$ and compute the characters in these subrepresentations.

3 Projective representation

Projective representations

Given a group G, a projective representation is a 'representation up to a phase':

$$X_g X_h = e^{i\omega(g,h)} X_{gh}. \tag{3}$$

Here, the ω's are called *2-cocycles* and satisfy the *2-cocycle constraint.*
A *gauge transformation* of the projective representation is a redefinition of the representation matrices of the form

$$X_g \mapsto e^{i\gamma(g)} X_g \tag{4}$$

for arbitrary phases $\gamma(g)$. A *non-trivial* projective representation is one for which not all the $\omega(g,h)$ can be made 0 by making use of gauge transformations.

3.4. The smallest group with a non-trivial projective representation is the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Check that a projective representation of this group is formed by the Pauli matrices:

$$D((0,0)) = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; D((0,1)) = \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

$$D((1,0)) = \sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; D((1,1)) = i\sigma_Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6}$$

and find the 2-cocycles.

Groups & Representations
Exercise session 5

April 29, 2022
16:00-17:30

1 Tensor product of representations

Tensor products

Given a d-dimensional vector space V with orthogonal basis $\{|\psi_i\rangle \, | i = 1, 2..., d\}$, the tensor product $V \otimes V$ is a d^2-dimensional vector space with orthogonal basis $\{|\psi_i\rangle \otimes |\psi_j\rangle \, | i, j = 1, 2..., d\}$. We will simply write $|\psi_i\rangle \otimes |\psi_j\rangle \equiv |\psi_i\rangle |\psi_j\rangle \equiv |\psi_{ij}\rangle$.

Given linear operators A and B acting on V, $A \otimes B$ acts on $V \otimes V$ according to $(A \otimes B) |\psi_{ij}\rangle = A |\psi_i\rangle \otimes B |\psi_j\rangle$.

If we are given some matrix representation of A and B, $\mathbf{A}$ and $\mathbf{B}$, with respect to the basis vectors $|\psi_i\rangle$, then the matrix representation of $A \otimes B$ with respect to the basis $\{|\psi_i\rangle \otimes |\psi_j\rangle \, | i, j = 1, 2..., d\}$ is given by the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \otimes \mathbf{B}$.

1.1. (a) Given a representation $\{U(g) | g \in \mathsf{G}\}$ of a (finite) group G, show that the tensor product $\{U(g) \otimes U(g) | g \in \mathsf{G}\}$ also forms a representation of this group. What is its dimension?

(b) What are the characters of the tensor product representation in function of the characters of the representation $\{U(g) | g \in \mathsf{G}\}$?

2 Characters of group representations

Decomposing reducible representations

Given a reducible representation, the central question of representation theory is which irreps this reducible representation contains and with which multiplicities. This question can be answered once the full character table of the group is known. Recall that the *character* of a group element $g \in \mathsf{G}$ in the representation a is defined as

$$\chi^{(a)}(g) := \mathrm{Tr}\Big(X_g^{(a)}\Big). \tag{1}$$

The character is clearly a *class function*, it only depends on the conjugacy class to which g belongs and is furthermore independent of the basis in which the $X_g^{(a)}$ are expressed, in other words: equivalent irreps have the same character.
Some important properties (extra exercise):

- $\chi^{(a\otimes b)}(g) = \chi^{(a)}(g) \cdot \chi^{(b)}(g)$,
- $\chi^{(a\oplus b)}(g) = \chi^{(a)}(g) + \chi^{(b)}(g)$.

Given representations a and b, not necessarily irreducible ones, we define the *inner product of characters* as[a]

$$\Big\langle \chi^{(a)} \Big| \chi^{(b)} \Big\rangle := \frac{1}{|\mathsf{G}|} \sum_g \overline{\chi}^{(a)}(g) \chi^{(b)}(g). \tag{2}$$

With respect to this inner product, the characters of the irreps of a group are orthonormal: $\langle \chi^{(\alpha)} | \chi^{(\beta)} \rangle = \delta_{\alpha,\beta}$ if α and β are irreps.
The number of times a given irrep α is contained in a reducible representation a, its multiplicity, is then given by $\langle \chi^{(\alpha)} | \chi^{(a)} \rangle$.

[a] Check that this indeed defines an inner product.

2.2. Verify that the chararacters of the irreps of $\mathbb{Z}_N$ are orthonormal. Hint: you can use $\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}(n-n')k} = \delta_{n,n'}$.

2.3. Recall the character table of S_3 from a previous exercise:

	E	$2C_3$	$3\sigma_V$
A_1	1	1	1
E	2	-1	0
A_2	1	1	-1

(3)

(a) Compute the character of $E \otimes E$ and decompose in irreps.

(b) Compute the character of $E \otimes E \otimes E$ and decompose in irreps.

(c) Compute the character of $E^{\otimes N}$, decompose in irreps and check for $N = 2, 3$ that you reobtain the previous results.

2.4. Recall the quaternion group Q from a previous exercise session: its 8 group elements are $\{\pm 1, \pm i, \pm j, \pm k\}$ and the multiplication is given by $i^2 = j^2 = k^2 = 1$, $ij = -ji = k$,

$jk = -kj = i$, $ki = -ik = j$. A faithful two-dimensional representation was given by the Pauli matrices:

$$\pm 1 \mapsto \pm \mathbb{1} \tag{4a}$$
$$\pm i \mapsto \pm i\sigma_X \tag{4b}$$
$$\pm j \mapsto \pm i\sigma_Y \tag{4c}$$
$$\pm k \mapsto \pm i\sigma_Z. \tag{4d}$$

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

Let us denote this representation by P.

(a) From the relation

$$\sum_\alpha \left(d^{(\alpha)}\right)^2 = |\mathsf{G}| \tag{6}$$

find out how many irreps there are and what their corresponding dimensions are.

(b) Construct a character table and fill in the characters of P.

(c) Explicitly find all one-dimensional irreps. Hint: begin by showing that all the characters of the one-dimensional irreps are ± 1.

(d) Decompose $P \otimes P$.

Groups & Representations
Exercise session 6

May 6, 2022
16:00-17:30

1 Characters of group representations

Decomposing reducible representations

Given a reducible representation, the central question of representation theory is which irreps this reducible representation contains and with which multiplicities. This question can be answered once the full character table of the group is known.
Recall that the *character* of a group element $g \in \mathsf{G}$ in the representation a is defined as

$$\chi^{(a)}(g) := \mathrm{Tr}\Big(X_g^{(a)}\Big). \tag{1}$$

The character is clearly a *class function*, it only depends on the conjugacy class to which g belongs and is furthermore independent of the basis in which the $X_g^{(a)}$ are expressed, in other words: equivalent irreps have the same character.
Some important properties (extra exercise):

- $\chi^{(a\otimes b)}(g) = \chi^{(a)}(g) \cdot \chi^{(b)}(g)$,
- $\chi^{(a\oplus b)}(g) = \chi^{(a)}(g) + \chi^{(b)}(g)$.

Given representations a and b, not necessarily irreducible ones, we define the *inner product of characters* as[a]

$$\Big\langle \chi^{(a)} \Big| \chi^{(b)} \Big\rangle := \frac{1}{|\mathsf{G}|} \sum_g \overline{\chi}^{(a)}(g) \chi^{(b)}(g). \tag{2}$$

With respect to this inner product, the characters of the irreps of a group are orthonormal: $\langle \chi^{(\alpha)} | \chi^{(\beta)} \rangle = \delta_{\alpha,\beta}$ if α and β are irreps.
The number of times a given irrep α is contained in a reducible representation a, its multiplicity, is then given by $\langle \chi^{(\alpha)} | \chi^{(a)} \rangle$.

[a] Check that this indeed defines an inner product.

1.1. Recall the quaternion group Q from a previous exercise session: its 8 group elements are $\{\pm 1, \pm i, \pm j, \pm k\}$ and the multiplication is given by $i^2 = j^2 = k^2 = 1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. A faithful two-dimensional representation was given by the Pauli matrices:

$$\pm 1 \mapsto \pm \mathbb{1} \tag{3a}$$
$$\pm i \mapsto \pm i\sigma_X \tag{3b}$$
$$\pm j \mapsto \pm i\sigma_Y \tag{3c}$$
$$\pm k \mapsto \pm i\sigma_Z. \tag{3d}$$

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Let us denote this representation by P.

(a) From the relation

$$\sum_\alpha \left(d^{(\alpha)}\right)^2 = |\mathsf{G}| \tag{5}$$

find out how many irreps there are and what their corresponding dimensions are.

(b) Construct a character table and fill in the characters of P.

(c) Explicitly find all one-dimensional irreps. Hint: begin by showing that all the characters of the one-dimensional irreps are ± 1.

(d) Decompose $P \otimes P$.

2 SU_2 and its Lie algebra $\mathfrak{su}(2)$

Lie algebras

A *Lie algebra* $\mathfrak{g}$ is a vector space, in our case over $F = \mathbb{R}, \mathbb{C}$, endowed with a *Lie bracket* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ that

(a) is bilinear $[\lambda x + y, z] = \lambda[x,z] + [y,z]$, $[x, \lambda y + z] = \lambda[x,y] + [x,z]$,

(b) is skewsymmetric $[x,y] = -[y,x]$,

(c) satisfies the Jacobi identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$.

In case we are dealing with associative algebras (eg. matrix algebras), we can define the Lie bracket to be the commutator $[x,y] := x \cdot y - y \cdot x$ and it will automatically satisfy the Jacobi identity (extra exercise).
The basis vectors of $\mathfrak{g}$, $\{T^i | i = 1, 2, ..., \dim_F \mathfrak{g}\}$ are often called the *generators* of $\mathfrak{g}$.
The Lie bracket is fully determined by its action on the generators and the *structure constants* f^{ab}_c are defined by $[T^a, T^b] = \sum_c f^{ab}_c T^c$.

A *Lie group* is a continuous group, ie. the group elements depend on a finite number of continuous parameters [a].
The Lie algebra $\mathfrak{g}$ of a Lie group G is spanned by infinitesimal group transformations close to the identity of the group (see exercises).

[a] For a more rigorous definition see the references listed on Ufora.

2.2. (a) Show that $\mathsf{SU}_N := \{M \in \mathbb{C}^{N\times N} | \det M = 1, M^\dagger = M^{-1}\}$ forms a group. It is called the *special unitary group*. 'Special' refers to the determinant being 1, 'unitary' because all matrices are unitary ($M^\dagger = M^{-1}$).

(b) Consider the exponential map that allows us to write every element $M \in \mathsf{SU}_N$ *close to the identity* as an exponential of its Lie algebra $\mathfrak{su}(N)$:

$$M = \exp\left(i \sum_a \theta_a T^a\right), \qquad \theta_a \in \mathbb{R}. \tag{6}$$

Impose $\det M = 1$ and $M^\dagger = M^{-1}$ to check which matrices $\mathfrak{su}(N)$ contains. What is the dimension of $\mathfrak{su}(N)$? Hint: use $\det \exp = \exp \operatorname{tr}$.

2.3. Consider SU_2.

(a) Consider a general 2×2 matrix

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \tag{7}$$

Impose $\det M = 1$ and $M^\dagger = M^{-1}$, conclude that M can be written as

$$M = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{8}$$

Count the number of free real parameters M has. Is this in accordance with what you found in the previous exercise?

(b) Show that every element of SU_2 can be written as

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = a\mathbb{1}_2 + ib\sigma_X + ic\sigma_Y + id\sigma_Z, \qquad a, b, c, d \in \mathbb{R}. \tag{9}$$

Find the parameters a, b, c, d in function of α and β.

(c) We denote $t_i := \frac{\sigma_i}{2}$ and $\sigma_{\vec{n}} := n_x t_x + n_y t_y + n_z t_z$, $\vec{n}$ being a real unit vector. Show that $(\sigma_{\vec{n}})^2 = \frac{\mathbb{1}_2}{4}$.

(d) Show that the SU_2 matrices can also be parametrized as

$$R_{\vec{n}}(\theta) = e^{i\theta\sigma_{\vec{n}}} = \cos\left(\frac{\theta}{2}\right)\mathbb{1}_2 + 2i\sin\left(\frac{\theta}{2}\right)\sigma_{\vec{n}}, \qquad \theta \in [0, 2\pi[\tag{10}$$

by writing $e^{i\theta\sigma_{\vec{n}}}$ as a Taylor expansion.

(e) Expand $R_{\vec{n}}(\delta\theta)$ to first order in $\delta\theta$, find the generators of the Lie algebra $\mathfrak{su}(2)$ and compute its commutation relations.

Groups & Representations
Exercise session 7

May 6, 2022
14:30-17:30

1 Representations of Lie algebras

Representations of Lie algebras

Given a Lie algebra $\mathfrak{g}$, a representation of $\mathfrak{g}$ is a (finite-dimensional) vector space V and a linear map $\rho : \mathfrak{g} \mapsto \mathsf{GL}(V)$ such that it is a *Lie algebra homomorphism*:

$$\rho([x,y]) = [\rho(x), \rho(y)], \tag{1}$$

the brackets on the right-hand side denoting the commutator of matrices.

1.1. In this exercise we construct the so called *adjoint representation* of a Lie algebra $\mathfrak{g}$. It is defined as

$$\mathrm{R_{ad}} : \mathfrak{g} \mapsto \mathsf{GL}(\mathfrak{g}), \qquad x \mapsto \mathrm{ad}_x, \tag{2}$$

where

$$\mathrm{ad}_x(y) = [x,y]. \tag{3}$$

(a) Show that

$$([\mathrm{ad}_x, \mathrm{ad}_y])(z) = \mathrm{ad}_{[x,y]}(z), \qquad \forall z \in \mathfrak{g}, \tag{4}$$

which demonstrates that $\mathrm{R_{ad}}$ satisfies (1).

(b) Given that

$$\left[T^a, T^b\right] = \sum_c f_c^{ab} T^c, \tag{5}$$

show that

$$\left(\mathrm{R_{ad}}\left(T^a\right)\right)^b{}_c = -f_c^{ab}. \tag{6}$$

(c) What will be the dimension of the adjoint representation of $\mathfrak{su}(2)$? Construct this representation.

2 SO_3 and its Lie algebra $\mathfrak{so}(3)$

2.2. Consider the exponential map that allows us to write every element $M \in \mathsf{SO}_N$ *close to the identity* as an exponential of its Lie algebra $\mathfrak{so}(N)$:

$$M = \exp\left(i \sum_a \theta_a T^a\right), \qquad \theta_a \in \mathbb{R}. \tag{7}$$

Impose $\overline{M} = M$[1], $\det M = 1$ and $M^\top = M^{-1}$ to check which matrices $\mathfrak{so}(N)$ contains. What is the dimension of $\mathfrak{so}(N)$? Hint: use $\det\exp = \exp\operatorname{tr}$.

2.3. Consider now SO_3. Every group element of SO_3 can be written as the product of three fundamental rotations, for example one around the x, y and z axis:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \tag{8a}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{8b}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8c}$$

(a) Expand each of these matrices to first order in $\delta\theta$, $R(\delta\theta) = \mathbb{1}_3 + i\delta\theta J_i$ and read of the generators $\{J_x, J_y, J_z\}$.

(b) Find the structure factors of $\mathfrak{so}(3)$ and show that $\mathfrak{so}(3) \cong \mathfrak{su}(2)$.

(c) Show that following operators *realise* $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, ie. satisfy the commutation relations of $\mathfrak{so}(3) \cong \mathfrak{su}(2)$:

$$\widehat{J}_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \tag{9a}$$

$$\widehat{J}_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \tag{9b}$$

$$\widehat{J}_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{9c}$$

Notice that these operators correspond to the components of the quantum mechanical operator $\widehat{\vec{L}} = -i\widehat{\vec{r}} \times \vec{\nabla}$. Which can be derived from the classical angular momentum vector $\vec{L} = \vec{r} \times \vec{p}$ by substituting $\vec{p} \mapsto -i\vec{\nabla}$.[2]

3 The 2-to-1 homomorphism $\mathsf{SU}_2 \mapsto \mathsf{SO}_3$

The 2-to-1 homomorphism $\mathsf{SU}_2 \mapsto \mathsf{SO}_3$

We have shown above that the *Lie algebras* of SU_2 and SO_3 are isomorphic. In this section we will examine the exact relation between the *Lie groups* SU_2 and SO_3. We will find that there exists a surjective homomorphism $\psi : \mathsf{SU}_2 \mapsto \mathsf{SO}_3$ that is moreover 2-to-1. More precisely, with every matrix $O(\theta_x, \theta_y, \theta_z) \in \mathsf{SO}_3$ we will be able to associate a matrix $U(\theta_x, \theta_y, \theta_z) \in \mathsf{SU}_2$ such that $\psi(U) = O$ and also $\psi(-U) = O$. This demonstrates that

$$\mathsf{SU}_2/\mathbb{Z}_2 \cong \mathsf{SO}_3, \tag{10}$$

where the normal $\mathbb{Z}_2$ subgroup of SU_2 is given by $\mathbb{Z}_2 = \{\mathbb{1}_2, -\mathbb{1}_2\}$. SU_2 is the so called *universal covering group* corresponding to the Lie algebra $\mathfrak{su}(2)$.

[1] Notice we did not have to do this in case of SU_2 as matrices in SU_2 are in general complex.
[2] We take $\hbar = 1$.

3.4. Consider some vector $\vec{x} \in \mathbb{R}^3$ and some real matrix $O \in \mathbb{R}^{3\times 3}$. Write $\vec{x}' = O\vec{x}$ and show that if $\vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y}$ this implies that $O^{-1} = O^\top$ and that if $\vec{x}' \times \vec{y}' \cdot \vec{z}' = \vec{x} \times \vec{y} \cdot \vec{z}$ then $\det O = 1$, hence $O \in \mathsf{SO}_3$. Hint: $\det A = \sum_{i_1,i_2,\ldots,i_n} \varepsilon_{i_1,i_2,\ldots,i_n} A_{1,i_1} A_{2,i_2} \ldots A_{n,i_n}$.

3.5. We construct ψ:

(a) Define $x := \sum_i x^i \sigma_i$, $\vec{x} \in \mathbb{R}^3$. This x is an element of $\mathfrak{su}(2)$!
Show that $\vec{x} \cdot \vec{y} = \frac{1}{2}\,\mathrm{Tr}(xy)$, that $\vec{x} \times \vec{y}$ corresponds to $\frac{1}{2i}[x, y]$ and $\vec{x} \times \vec{y} \cdot \vec{z} = \frac{1}{4i}\,\mathrm{Tr}([x, y]z)$.

(b) For $U \in \mathsf{SU}_2$, show that $\mathrm{Tr}\big(UxU^{-1}\big) = 0$ and $(UxU^{-1})^\dagger = UxU^{-1}$. Hence, $UxU^{-1} \in \mathfrak{su}(2)$.

(c) Now we show that the map induced by U, $O := \psi(U) : x \mapsto UxU^{-1}$, is in fact a special orthogonal transformation, $O \in \mathsf{SO}_3$. Concretely: $UxU^{-1} = \sum_i \tilde{x}^i \sigma_i$, where $\tilde{\vec{x}} = O\vec{x}$.
Show that

i. $\vec{x} \cdot \vec{y}$ remains invariant under O,

ii. same for $\vec{x} \times \vec{y} \cdot \vec{z}$.

(d) Now show that this map is a homomorphism: $\psi(U_1)\psi(U_2) = \psi(U_1 U_2)$.

(e) Now we need to show that ψ is surjective. We leave this as an extra exercise as this is quiet tedious to show. One approach to prove this, is to find explicit SU_2 matrices that get mapped to the generators of SO_3 (8a-8c).

(f) Observe that $\psi(U) = \psi(-U)$ and show that the kernel of ψ is $\mathbb{Z}_2 \cong \{\pm \mathbb{1}_2\}$.

(g) Conclude that $\mathsf{SU}_2/\mathbb{Z}_2 \cong \mathsf{SO}_3$. Hint: use that $\mathsf{G}_1/\ker\varphi \cong \mathrm{im}\varphi$, where $\varphi : \mathsf{G}_1 \mapsto \mathsf{G}_2$ is a homomorphism (to prove this is an extra exercise).

Groups & Representations
Exercise session 8

May 20, 2022
16:00-17:30

1 Tensor product of representations of Lie algebras

1.1. Suppose (ρ_1, V_1) and (ρ_2, V_2) are representations of a Lie algebra $\mathfrak{g}$. Show that

$$(\rho_1 \otimes \rho_2)(X) = \rho_1(X) \otimes \mathbb{1} + \mathbb{1} \otimes \rho_2(X) \tag{1}$$

defines a representation on $V_1 \otimes V_2$, the *tensor product representation* of (ρ_1, V_1) and (ρ_2, V_2), denoted by $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$.

2 Tensor product of representations of $\mathfrak{su}(2)$

Tensor product of representations of $\mathfrak{su}(2)$

We will label the highest weight representations by their *spin label* j, which is related to the Λ in the book of Fuchs and Schweigert by $j = \Lambda/2$. We will also denote the representations by their dimension, ie. $j \leftrightarrow \mathbf{2j+1} \leftrightarrow \mathbf{\Lambda + 1}$.

Three particularly important representations of $\mathfrak{su}(2)$:

- The *trivial* representation, $j = 0$, $\Lambda = 0$, $\mathbf{1}$, *singlet*, *scalar*.
- The *fundamental* representation, $j = 1/2$, $\Lambda = 1$, $\mathbf{2}$, *doublet*, *spinor*.
- The *adjoint* representation, $j = 1$, $\Lambda = 2$, $\mathbf{3}$, *triplet*, *vector*.

Recall that the tensor product of representations of $\mathfrak{su}(2)$ decomposes as

$$j_1 \otimes j_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} j. \tag{2}$$

2.1. Show that $\dim j_1 \otimes j_2 = \dim \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} j$.

Hint: assume w.l.o.g $j_1 \geq j_2$ and observe that $\dim \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} j$ is an arithmetic sequence.

2.2. Compute the tensor product of two spin 1/2 representations and of three spin 1/2 representations.

2.3. Show that every highest weight representation of $\mathfrak{su}(2)$ can be obtained by taking tensor products of the *fundamental representation* $\mathbf{2}$.

2.4. In this exercise we will numerically investigate the tensor product representation $\mathbf{2} \otimes \mathbf{2}$.

(a) Consider the adjoint (spin 1) representation of $\mathfrak{su}(2)$ from the previous exercise session:

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tag{3a}$$

$$J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tag{3b}$$

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3c}$$

Go to the so called spherical basis $R_i = U J_i U^\dagger$ with the basis transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}. \tag{4}$$

(b) Define an arbitrary element of $\mathfrak{su}(2)$ in the spin 1/2 representation: $F = x\frac{\sigma_X}{2} + y\frac{\sigma_Y}{2} + z\frac{\sigma_Z}{2}$, with arbitrary $x, y, z \in \mathbb{R}$.

(c) Construct the tensor product representation $\mathbf{2} \otimes \mathbf{2}$, $F \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes F$.

(d) Implement the basis transformation

$$V = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

and show that in this basis the tensor product representation falls apart in a trivial block and a three-dimensional representation, ie. $V(F \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes F)V^\dagger = 0 \oplus A$. Where the three-dimensional adjoint representation reads $A = xR_x + yR_y + zR_z$.