Groups & Representations
Exercise session 1 - Solutions

February 25, 2022
14:30-17:30

1 Symmetries in classical and quantum systems

1.1. *Consider a Hamiltonian system described by the (independent) coordinates* $\{q^i, p_i | i = 1, 2, ..., N\}$ *and a generic Hamiltonian* $H \equiv H(q^i, p_i)$. *Recall that the Hamilton equations governing the dynamics of such a system read*

$$\begin{cases} \dot{q}^i &= \frac{\partial H}{\partial p_i} \qquad i = 1, 2, ..., N \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} \qquad i = 1, 2, ..., N \end{cases}. \tag{1}$$

(a) Define $\vec{\eta} := (q^1, ..., q^N, p_1, ..., p_N)$ *to write the Hamilton equations as*

$$\dot{\eta}^i = \sum_j \Omega_{ij} \frac{\partial H}{\partial \eta^j}, \tag{2}$$

where $\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$.

(b) Consider a coordinate transformation of the form

$$\begin{cases} q^i & \mapsto Q^i\left(q^i, p_i\right) \\ p_i & \mapsto P_i\left(q^i, p_i\right) \end{cases}. \tag{3}$$

Which coordinates transformations preserve the form of the Hamilton equations? Such coordinate transformations are called canonical coordinate transformations.

Solution:

(a) Trivial

(b) Defining the coordinate $\vec{\tilde{\eta}} := \left(Q^1, ..., Q^N, P_1, ..., P_N\right)$, we can use the chain rule to write the left-hand side of (2) as

$$\dot{\eta}^i = \sum_j \frac{\partial \eta^i}{\partial \tilde{\eta}^j} \dot{\tilde{\eta}}^j, \tag{4}$$

whereas the right-hand side can be written as

$$\sum_j \Omega_{ij} \frac{\partial H}{\partial \eta^j} = \sum_{j,k} \Omega_{ij} \frac{\partial H}{\partial \tilde{\eta}^k} \frac{\partial \tilde{\eta}^k}{\partial \eta^j}. \tag{5}$$

Hence, (2) can be written as

$$\sum_j \frac{\partial \eta^i}{\partial \tilde{\eta}^j} \dot{\tilde{\eta}}^j = \sum_{j,k} \Omega_{ij} \frac{\partial H}{\partial \tilde{\eta}^k} \frac{\partial \tilde{\eta}^k}{\partial \eta^j} \tag{6}$$

$$\implies \sum_{i,j} \frac{\partial \tilde{\eta}^l}{\partial \eta^i} \frac{\partial \eta^i}{\partial \tilde{\eta}^j} \dot{\tilde{\eta}}^j = \sum_{i,j,k} \frac{\partial \tilde{\eta}^l}{\partial \eta^i} \Omega_{ij} \frac{\partial H}{\partial \tilde{\eta}^k} \frac{\partial \tilde{\eta}^k}{\partial \eta^j} \tag{7}$$

$$\implies \dot{\tilde{\eta}}^l = \sum_{i,j,k} \frac{\partial \tilde{\eta}^l}{\partial \eta^i} \Omega_{ij} \frac{\partial H}{\partial \tilde{\eta}^k} \frac{\partial \tilde{\eta}^k}{\partial \eta^j}. \tag{8}$$

Defining the (inverse) Jacobian matrix $M_{ij} := \frac{\partial \tilde{\eta}^j}{\partial \eta^i}$, and relabeling indices allow us to write (8) as

$$\dot{\tilde{\eta}}^i = \sum_{j,k,l} M_{ji} \Omega_{jk} M_{kl} \frac{\partial H}{\partial \tilde{\eta}^l}. \tag{9}$$

Comparing with (2), it is clear that the form of the Hamilton equations is preserved if the Jacobian matrix satisfies

$$\sum_{j,k} M_{ji} \Omega_{jk} M_{kl} = \Omega_{i,l}, \tag{10}$$

or in matrix notation:

$$\boxed{M^\top \Omega M = \Omega}. \tag{11}$$

Matrices that satisfy this property are so-called *symplectic matrices*. These matrices form a group, the so called *symplectic group*, that we shall denote by $\mathsf{Sp}\left(2n, \mathbb{R}\right) := \left\{M \in \mathbb{R}^{2n \times 2n} | M^\top \Omega M = \Omega\right\}$. In other words, a coordinate transformation $\vec{\eta} \mapsto \tilde{\vec{\eta}}$ is canonical, ie. preserves the form of the Hamilton equations, if the Jacobian associated to this coordinate transformation is a symplectic matrix.[1]

1.2. *Consider a Hamiltonian system as in the previous exercise. The* Poisson bracket *on differentiable functions $f\left(q^i, p_i\right)$ is defined as*

$$\{f, g\} := \sum_i \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \left(p_i \leftrightarrow q^i\right). \tag{12}$$

(a) *Compute:*

  i. $\left\{q^i, q^j\right\}$,

  ii. $\left\{p_i, p_j\right\}$,

  iii. $\left\{q^i, p_j\right\}$.

  How does this relate to the canonical commutation relations encountered in quantum mechanics?

(b) *Show that the Poisson bracket is:*

[1]Unfortunately, there are different ways in which the symplectic group is denoted in the literature. For example, sometimes $\mathsf{Sp}\left(2, \mathbb{R}\right)$ is written to denote $\mathsf{Sp}\left(2n, \mathbb{R}\right)$. The symplectic group (over the complex numbers) $\mathsf{Sp}\left(2n, \mathbb{C}\right)$ should also not be confused with the *compact symplectic group* $\mathsf{USp}\left(2n\right) := \mathsf{Sp}\left(2n, \mathbb{R}\right) \cap \mathsf{U}_{2n}$, which is also sometimes denoted as $\mathsf{Sp}(2n), \mathsf{Sp}(n), \ldots$

i. skew-symmetric,

ii. bilinear,

and satisfies the

iii. Jacobi identity

$$\{f, \{g, h\}\} + (cyclic) = 0. \tag{13}$$

The space of functions $f\left(q^i, p_i\right)$ on phase space can be endowed with the structure of a vector space. A vector space that comes with a bilinear map that satisfies the Jacobi identity and is skew-symmetric, is called a Lie algebra, *and plays a very important role in - among many other things - the classification of continuous groups, infinitesimal symmetry transformations in field theory,... Different Lie algebras will be encountered many times.*

(c) *As a concrete example, consider a three-dimensional system described by Cartesian coordinates x^1, x^2, x^3. The angular momentum (pseudo-)vector $\vec{L}$ is defined as $\vec{L} := \vec{r} \times \vec{p}$. Compute $\{L_i, L_j\}$.*

(d) *Show that the Poisson bracket is preserved under canonical transformations. Hint: write $\{f, g\} = \sum_{i,j} \frac{\partial f}{\partial \eta^i} \Omega_{ij} \frac{\partial g}{\partial \eta^j}$.*

Solution:

(a) From the definition of the Poisson bracket it immediately follows that

i. $\left\{q^i, q^j\right\} = 0$,

ii. $\{p_i, p_j\} = 0$,

iii. $\left\{q^i, p_j\right\} = \delta^i_j$.

These Poisson brackets are similar to the canonical commutation relations that you know from quantum mechanics:

$$\begin{cases} \left[\hat{x}^i, \hat{x}^j\right] & = 0 \\ \left[\hat{p}_i, \hat{p}_j\right] & = 0 \\ \left[\hat{x}^i, \hat{p}_j\right] & = i\hbar\delta^i_j \end{cases} . \tag{14}$$

This suggests that quantum mechanics can be obtained from (classical) Hamiltonian mechanics by promoting the (functions of) q^i, p_i to operators and replacing Poisson brackets by commutators according to $i\hbar \{\cdot, \cdot\}$ " = " $[\cdot, \cdot]$.

(b)
- Follows immediately from the definition of the Poisson bracket.
- Idem.
- A very straightforward but tedious computation results in the following:

$$\{f, \{g, h\}\} \tag{15}$$

$$=: \sum_i \frac{\partial f}{\partial q^i} \frac{\partial \{g, h\}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \{g, h\}}{\partial q^i} \tag{16}$$

$$=: \sum_{i,j} \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \left(\frac{\partial g}{\partial q^j} \frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial p_j} \frac{\partial h}{\partial q^j} \right) - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} \left(\frac{\partial g}{\partial q^j} \frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial p_j} \frac{\partial h}{\partial q^j} \right) \tag{17}$$

$$\begin{aligned} = \sum_{i,j} & \frac{\partial f}{\partial q^i} \frac{\partial^2 g}{\partial p_i \partial q^j} \frac{\partial h}{\partial p_j} + \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} \frac{\partial^2 h}{\partial p_i \partial p_j} - \frac{\partial f}{\partial q^i} \frac{\partial^2 g}{\partial p_i \partial p_j} \frac{\partial h}{\partial q^j} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_j} \frac{\partial^2 h}{\partial p_i \partial q^j} \\ & - \frac{\partial f}{\partial p_i} \frac{\partial^2 g}{\partial q^i \partial q^j} \frac{\partial h}{\partial p_j} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^j} \frac{\partial^2 h}{\partial q^i \partial p_j} + \frac{\partial f}{\partial p_i} \frac{\partial^2 g}{\partial q^i \partial p_j} \frac{\partial h}{\partial q^j} + \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j} \frac{\partial^2 h}{\partial q^i \partial q^j}. \end{aligned} \tag{18}$$

The other two terms can be found by replacing $f \mapsto g, g \mapsto h, h \mapsto f$ and $f \mapsto h, g \mapsto f, h \mapsto g$:

$$\{g,\{h,f\}\} \tag{19}$$
$$=: \sum_i \frac{\partial g}{\partial q^i}\frac{\partial \{h,f\}}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial \{h,f\}}{\partial q^i} \tag{20}$$
$$=: \sum_{i,j} \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i}\left(\frac{\partial h}{\partial q^j}\frac{\partial f}{\partial p_j} - \frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q^j}\right) - \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i}\left(\frac{\partial h}{\partial q^j}\frac{\partial f}{\partial p_j} - \frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q^j}\right) \tag{21}$$
$$= \sum_{i,j} \frac{\partial g}{\partial q^i}\frac{\partial^2 h}{\partial p_i \partial q^j}\frac{\partial f}{\partial p_j} + \frac{\partial g}{\partial q^i}\frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial p_i \partial p_j} - \frac{\partial g}{\partial q^i}\frac{\partial^2 h}{\partial p_i \partial p_j}\frac{\partial f}{\partial q^j} - \frac{\partial g}{\partial q^i}\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial p_i \partial q^j}$$
$$- \frac{\partial g}{\partial p_i}\frac{\partial^2 h}{\partial q^i \partial q^j}\frac{\partial f}{\partial p_j} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial q^i \partial p_j} + \frac{\partial g}{\partial p_i}\frac{\partial^2 h}{\partial q^i \partial p_j}\frac{\partial f}{\partial q^j} + \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial q^i \partial q^j}. \tag{22}$$

$$\{h,\{f,g\}\} \tag{23}$$
$$=: \sum_i \frac{\partial h}{\partial q^i}\frac{\partial \{f,g\}}{\partial p_i} - \frac{\partial h}{\partial p_i}\frac{\partial \{f,g\}}{\partial q^i} \tag{24}$$
$$=: \sum_{i,j} \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j}\right) - \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i}\left(\frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j}\right) \tag{25}$$
$$= \sum_{i,j} \frac{\partial h}{\partial q^i}\frac{\partial^2 f}{\partial p_i \partial q^j}\frac{\partial g}{\partial p_j} + \frac{\partial h}{\partial q^i}\frac{\partial f}{\partial q^j}\frac{\partial^2 g}{\partial p_i \partial p_j} - \frac{\partial h}{\partial q^i}\frac{\partial^2 f}{\partial p_i \partial p_j}\frac{\partial g}{\partial q^j} - \frac{\partial h}{\partial q^i}\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial p_i \partial q^j}$$
$$- \frac{\partial h}{\partial p_i}\frac{\partial^2 f}{\partial q^i \partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial q^j}\frac{\partial^2 g}{\partial q^i \partial p_j} + \frac{\partial h}{\partial p_i}\frac{\partial^2 f}{\partial q^i \partial p_j}\frac{\partial g}{\partial q^j} + \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial q^i \partial q^j}. \tag{26}$$

So indeed all terms cancel.

(c) Even though there are $3 \times 3 = 9$ cases to consider, only 3 cases yield non-zero results and are independent since the Poisson bracket is skew-symmetric.
The components of $\vec{L}$ are:

$$\begin{cases} L_1 &= x^2 p_3 - x^3 p_2 \\ L_2 &= x^3 p_1 - x^1 p_3 \\ L_3 &= x^1 p_2 - x^2 p_1 \end{cases}. \tag{27}$$

Let us work one case out explicitly:

$$\{L_2, L_3\} = \{x^3 p_1 - x^1 p_3, x^1 p_2 - x^2 p_1\}. \tag{28}$$

We compute that

$$\{x^3 p_1, x^1 p_2\} = \sum_j \frac{\partial (x^3 p_1)}{\partial x^j}\frac{\partial (x^1 p_2)}{\partial p_j} - \sum_j \frac{\partial (x^3 p_1)}{\partial p_j}\frac{\partial (x^1 p_2)}{\partial x^j} = 0 - x^3 p_2 \tag{29}$$
$$\{x^3 p_1, x^2 p_1\} = 0 \tag{30}$$
$$\{x^1 p_3, x^1 p_2\} = 0 \tag{31}$$
$$\{x^1 p_3, x^2 p_1\} = x^2 p_3. \tag{32}$$

Hence, from bilinearity it follows that

$$\{L_2, L_3\} = x^2 p_3 - x^3 p_2 =: L_1. \tag{33}$$

For the other cases we similarly find that

$$\{L_1, L_2\} = L_3 \tag{34}$$
$$\{L_3, L_1\} = L_2. \tag{35}$$

We can summarize this result by writing

$$\boxed{\{L_i, L_j\} = \sum_{k=1}^{3} \varepsilon_{ijk} L_k}. \tag{36}$$

Notice that this compact expression covers all nine cases.
Also notice that since the set $\{L_1, L_2, L_3\}$ is closed under taking Poisson brackets (meaning that the Poisson bracket of two of these components yields another component or 0), $\{L_1, L_2, L_3\}$ forms in itself a Lie algebra (see above). Later we will see that the quantum version of (36), $\left[\hat{L}_i, \hat{L}_j\right] = i\hbar \sum_k \varepsilon_{ijk} \hat{L}_k$, constitutes the Lie algebra $\mathfrak{su}(2)$.

1.3. *Consider a quantum mechanical system that evolves in time according to some time-independent Hamiltonian $\widehat{H}$, show that the expectation value of every time-independent operator that commutes with $\widehat{H}$ is conserved in time.*

Let us call the operator that commutes with the Hamiltonian $\widehat{O}$, $\left[\widehat{H}, \widehat{O}\right] = 0$.

Recall from your course on quantum mechanics that in the Schrödinger picture the time evolution of some initial state $|\Psi_0\rangle$ is formally given by

$$|\Psi(t)\rangle = \exp\left(-\frac{i\widehat{H}t}{\hbar}\right) |\Psi_0\rangle . \tag{37}$$

Hence, it follows that

$$\left\langle \widehat{O} \right\rangle =: \langle \Psi(t)|\widehat{O}|\Psi(t)\rangle \tag{38}$$
$$= \langle \Psi_0| e^{\frac{i\widehat{H}t}{\hbar}} \widehat{O} e^{-\frac{i\widehat{H}t}{\hbar}} |\Psi_0\rangle \tag{39}$$
$$= \langle \Psi_0|\widehat{O} \cancel{e^{\frac{i\widehat{H}t}{\hbar}}} \cancel{e^{-\frac{i\widehat{H}t}{\hbar}}} |\Psi_0\rangle \tag{40}$$
$$= \langle \Psi_0|\widehat{O}|\Psi_0\rangle , \tag{41}$$

and since $|\Psi_0\rangle$ and $\widehat{O}$ are time independent, we conclude that

$$\boxed{\left[\widehat{H}, \widehat{O}\right] = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \widehat{O} \right\rangle = 0}. \tag{42}$$

2 Basic properties of groups

2.1. *Which of the following examples are (possibly infinite) groups? What is the order of the groups you find? Are they abelian or not?*

(a) The integers $\{0, 1, ..., N-1\}$ *with group multiplication addition modulo* N*.*

(b) The set of natural numbers $\mathbb{N}$ *with group multiplication addition.*

(c) The integers $\mathbb{Z}$ *with group multiplication addition.*

(d) The permutations of a set of N *elements with group multiplication the composition of functions.*

(e) $\mathsf{SO}_n := \left\{M \in \mathbb{R}^{n\times n} | \det M = 1, M^{-1} = M^\top\right\}$ *with group multiplication the ordinary matrix multiplication.*

To check whether these examples are groups we need to verify for every example that the multiplication is closed and associative, that there is a unit for the multiplication and that every element has an inverse (that is also contained in the group (!)).

We find that only (b) is not a group:

	Closed?	Unit?	Associative?	Inverse?	Order?	Abelian?
(a)	✓	0	✓	$N - x$	N	✓
(b)	✓	0	✓	No!		
(c)	✓	0	✓	$-x$	∞	✓
(d)	✓	id	✓	Inverse permutation	$N!$	No if $N > 2$
(e)	✓	$\mathbb{1}_n$	✓	M^{-1} (matrix inverse)	∞ if $n > 1$	No if $n > 2$

Some important comments:

- The first group, the *additive group of integers modulo* N or *cyclic group of order* N, is often denoted by $\mathbb{Z}_N$ or C_N. In physics, $\mathbb{Z}_2$ is of particular importance. Its multiplication table reads

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\,. \tag{43}$$

- The *additive group of integers* is often denoted by $(\mathbb{Z}, +)$ or simply as $\mathbb{Z}$.
- id denotes the identity permutation, ie. not permuting anything
- The *permutation group of* N *elements* is often denoted by S_N.
- SO_n is called the *special orthogonal group*. The adjective *special* refers to $\det M = 1$, whereas *orthogonal* refers to $M^\top = M^{-1}$. SO_n describes the proper (ie. without reflection) rotations in n spatial dimensions and is therefore also sometimes called the *rotation group in* n *dimension*. Including reflections results in the *(general) orthogonal group* $\mathsf{O}_n := \left\{M \in \mathbb{R}^{n\times n} || \det M| = 1, M^{-1} = M^\top\right\}$. Check that O_n is a subgroup of SO_n!
- It is important to check that the inverse of a group element is also contained in the group. In the example of SO_n this is not immediately clear. So concretely we need to check that if $M \in \mathsf{SO}_n$ then also $M^{-1} \in \mathsf{SO}_n$. And indeed, $\det M^{-1} = (\det M)^{-1} = 1$, $\left(M^{-1}\right)^\top = \left(M^\top\right)^{-1} = \left(M^{-1}\right)^{-1}$, where we used that the transpose of the inverse matrix is the inverse of the transpose matrix.

- In a previous course you saw (this is also an extra exercise) that every matrix in SO_2 can be parameterized as

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in [0, 2\pi). \tag{44}$$

  From this it immediately follows that SO_2 is of infinite order and abelian. Since for every $n > 2$ SO_n contains at least one SO_2 subgroup [2], it immediately follows that also SO_n for $n > 2$ is of infinite order. SO_1 is the trivial group containing only one element, namely 1.

- It is easy to see that SO_3 is non-abelian (try to find two rotations that do not commute). By a similar argument as before it follows that also SO_n $n > 3$ is therefore non-abelian.

[2]One SO_2 subgroup of $\mathsf{SO}_n, n > 2$ is for example (check this!):

$$\left\{ \left(\begin{array}{c|c} R(\theta) & 0 \\ \hline 0 & \mathbb{1}_{n-2} \end{array} \right) \middle| \ \theta \in [0, 2\pi) \right\} \leq \mathsf{SO}_n. \tag{45}$$

Groups & Representations
Exercise session 2 - Solutions

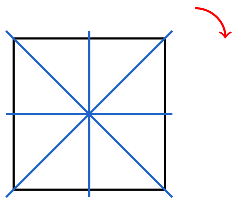
March 11, 2022
14:30-17:30

1 Basic properties of groups

1.1. *Construct* D_4. *Hint: recall from the lectures that* D_4 *is symmetry group of a regular 4-gon, aka. a square.*

Solution:

The symmetries of a square consist of doing nothing (identity), four reflections and clockwise rotations over 90°, 180° and 270°:



Hence, the order of D_4 is eight. The group is generated by two group elements, call them a and b, which can be taken to be a rotation over 90° clockwise and a reflection around the horizontal axis respectively (other choices such as a reflection around a diagonal axis are possible), with the constraints that $a^4 = e$ (rotating over 90° four times is equivalent to leaving the square untouched), $b^2 = e$ (doing a reflection twice is equivalent to doing nothing) and $aba = b$ (conjugating the reflection with a 90° rotation is equivalent to a simple reflection). Indeed, the three non-trivial rotations can be found by doing subsequent rotations over 90°, the vertical rotation by combining a 180° rotation with a horizontal reflection and a diagonal reflection by combining a rotation over 270° or 90° with a horizontal reflection.

In this way, the Cayley table is found to be:

	e	a	a^2	a^3	b	ba	ba^2	ba^3
e	e	a	a^2	a^3	b	ba	ba^2	ba^3
a	a	a^2	a^3	e	ba^3	b	ba	ba^2
a^2	a^2	a^3	e	a	ba^2	ba^3	b	ba
a^3	a^3	e	a	a^2	ba	ba^2	ba^3	b
b	b	ba	ba^2	ba^3	e	a	a^2	a^3
ba	ba	ba^2	ba^3	b	a^3	e	a	a^2
ba^2	ba^2	ba^3	b	ba	a^2	a^3	e	a
ba^3	ba^3	b	ba	ba^2	a	a^2	a^3	e

. (1)

A particularly simple way to define this group is as follows:

$$\mathsf{D}_4 = \left\langle a, b | a^4 = b^2 = (ab)^2 = e \right\rangle . \tag{2}$$

This notation is sometimes called a *presentation* of the group.[1] Its meaning is the following: every group element of D_4 can be written as the multiplication of the generators a and b and products of those products,... with the additional constraints $a^4 = b^2 = (ab)^2 = e$. You can check that in this way indeed 8 group elements are found: $e, a, a^2, a^3, b, ba, ba^2, ba^3$. Every other product of the generators is equal to one of these eight (e.g.: $bab = bb^{-1}a^{-1} = bba^3 = ea^3 = a^3$).

1.2. *Given some finite group G, show that every row and column of the Cayley table contains every group element exactly once.*

Solution:

Consider the row labeled by some fixed $g \in \mathsf{G}$. Suppose that this row contains the same element two times, in other words, there are $h_1, h_2 \in \mathsf{G}$, $h_1 \neq h_2$ such that $gh_1 = gh_2$. Since g has an inverse element, we can multiply $gh_1 = gh_2$ from the left with g^{-1} to obtain that $h_1 = h_2$, in contradiction with the assumption that $h_1 \neq h_2$, hence, every row contains every element at most once. Since every row contains $|\mathsf{G}|$ elements, it thus contains every group element exactly once.

Similarly for the columns.

You can check this explicitly for the multiplication table of D_4 above!

1.3. (a) *Given some finite group G, prove that for every $g \in \mathsf{G}$ the order of g is smaller than or equal to the order of the group. Hint: look at the set $\{1, g, g^2, ...\}$.*

(b) *Argue that every finite group has a cyclic subgroup.*

(c) *Argue that the order of every group element is a divisor of the order of the group.*

Solution:

(a) Let us first show that every group element of a finite group is of finite order. Consider therefore $1, g, g^2, g^3, ...$. Since the group is finite, there should exist m, k, $m > k$, such that $g^m = g^k$. Therefore $g^n = 1$ for $n := m - k$ finite.

Now we show that the order of every group element is at most $|\mathsf{G}|$. Consider the smallest n such that $g^n = 1$ and assume that $n > |\mathsf{G}|$. Consider the n group elements $1, g, g^2, ..., g^{n-1}$. These are all different. Indeed, suppose $g^{n_1} = g^{n_2}$ for certain $1 \leq n_1 < n_2 \leq n - 1$, then $g^{n_2 - n_1} = 1$, which would be in contradiction with the assumption that n is the smallest number that satisfies $g^n = 1$. However, that there are $n > |\mathsf{G}|$ group elements $1, g, g^2, ..., g^{n-1}$ that are all different can not be true as the group G has only $|\mathsf{G}|$ distinct elements. Hence, our initial assumption that $n > |\mathsf{G}|$ is false. □

(b) The trivial group $\mathsf{G} = \{e\}$ is by definition cyclic and thus trivially has a cyclic subgroup, namely itself.

Suppose $|\mathsf{G}| > 1$, every group element $g \neq 1$ gives rise to the cyclic group of order the order of g and has group elements $1, g, g^2, ..., g^{n-1}$.

As an example consider D_4 from above: the rotation over 90° generates a $\mathbb{Z}_4$ subgroup since the order of the 90° rotation is 4, every reflection is of order 2 and hence gives rise to a $\mathbb{Z}_2$ subgroup of D_4. Another $\mathbb{Z}_2$ subgroup is generated by the 180° rotation. Can you find other subgroups?

[1] In general one can write $\mathsf{D}_n = \left\langle a, b | a^n = b^2 = (ab)^2 = e \right\rangle$.

(c) Consider $\{1, g, g^2, ..., g^{n-1}\}$, n being the order of g. This forms a cyclic subgroup of G. From Lagrange's theorem it follows that the order of this subgroup divides the order of G. Hence n divides $|\mathsf{G}|$.

1.4. *Show there are two groups of order four (up to isomorphism). Hint: use (1.3c).*

Solution:

We label the distinct groups elements as $\{1, \sigma, \tau, \rho\}$ (there is always the unit element 1). In the previous exercise we saw that every group element is of order 2 or 4 (the only group element of order 1 is of course the identity, which is unique).

Suppose that the order of any group element (except for the identity), say σ, is 4, then necessarily $\tau = \sigma^2$ and $\tau = \sigma^3$. This corresponds to the group $\mathbb{Z}_4 \cong \{1, \sigma, \sigma^2, \sigma^3\} = \langle \sigma | \sigma^4 = 1 \rangle$.

If there is a group element of order 2, say $\sigma^2 = 1$, then all group elements (except for the identity) are of order 2, because if there were a group element of order 4, we would have the group $\mathbb{Z}_4$, in contradiction with the assumption that there is a group element of order 2.

We can also conclude that $\sigma\tau = \rho$ because $\sigma\tau = \sigma$ would imply that $\tau = 1$ and $\sigma\tau = \tau$ would imply that $\sigma = 1$, both in contradiction with the fact that all group elements are distinct. Similarly we can conclude $\rho = \tau\sigma$.

Hence, the group we find can be written as

$$\langle \sigma, \tau | \sigma^2 = \tau^2 = 1, \sigma\tau = \tau\sigma \rangle . \tag{3}$$

This is exactly the group $\mathbb{Z}_2 \times \mathbb{Z}_2$! The first factor $\mathbb{Z}_2$ corresponds to the $\mathbb{Z}_2$ subgroup $\{1, \sigma\}$, the second to the $\mathbb{Z}_2$ subgroup $\{1, \tau\}$.

An alternative solution would be to construct all the valid Cayley tables. (1.2) heavily constrains the possibilities and together with the axioms of finite groups also leads to these two possibilities.

2 Automorphisms

Group homomorphisms and the automorphism group

A *group homomorphism* φ is a map between groups $\varphi : \mathsf{G}_1 \mapsto \mathsf{G}_2$ that is compatible with the group structure, ie. $\varphi(g_1)\varphi(g_2) = \varphi(g_{12})$. A bijective group homomorphism is called a *group isomorphism*. A *group automorphism* is a group isomorphism from a group to itself.

2.5. Compute following automorphism groups

(a) $\mathsf{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$

(b) $\mathsf{Aut}(\mathbb{Z}_8)$

Solution:

(a) Since a homomorphism maps the identity to the identity, every automorphism is some permutation of the non-trivial group elements $(1,0), (0,1)$ and $(1,1)$. There are six such permutations:

$$
\begin{aligned}
(1,0) &\overset{\text{id}}{\mapsto} (1,0) & \quad (1,0) &\overset{\beta}{\mapsto} (0,1)\\
(0,1) &\mapsto (0,1) & (0,1) &\mapsto (1,0)\\
(1,1) &\mapsto (1,1) & (1,1) &\mapsto (1,1)
\end{aligned}
$$

$$
\begin{aligned}
(1,0) &\overset{\alpha}{\mapsto} (0,1) & \quad (1,0) &\overset{\beta\alpha}{\mapsto} (1,0)\\
(0,1) &\mapsto (1,1) & (0,1) &\mapsto (1,1)\\
(1,1) &\mapsto (1,0) & (1,1) &\mapsto (0,1)
\end{aligned}
$$

$$
\begin{aligned}
(1,0) &\overset{\alpha^2}{\mapsto} (1,1) & \quad (1,0) &\overset{\beta\alpha^2}{\mapsto} (1,1)\\
(0,1) &\mapsto (1,0) & (0,1) &\mapsto (0,1)\\
(1,1) &\mapsto (0,1) & (1,1) &\mapsto (1,0).
\end{aligned}
$$

To find which of these permutations are actually automorphisms, we need to check which permutations (besides the trivial permutation) are homomorphisms.
Let us consider α. We thus need to check that $\alpha(g_1g_2) = \alpha(g_1)\alpha(g_2)$ for all $g_1, g_2 \in \mathbb{Z}_2^{\times 2}$. A tedious but very easy computation results in:

$$\alpha((1,0)(1,0)) = \alpha((0,0)) = (0,0) = (0,1)(0,1) = \alpha((1,0))\alpha((1,0)) \tag{4}$$
$$\alpha((1,0)(0,1)) = \alpha((1,1)) = (1,0) = (0,1)(1,1) = \alpha((1,0))\alpha((0,1)) \tag{5}$$
$$\alpha((1,0)(1,1)) = \alpha((0,1)) = (1,1) = (0,1)(1,0) = \alpha((1,0))\alpha((1,1)) \tag{6}$$
$$\alpha((0,1)(0,1)) = \alpha((0,0)) = (0,0) = (1,1)(1,1) = \alpha((0,1))\alpha((0,1)) \tag{7}$$
$$\alpha((0,1)(1,1)) = \alpha((1,0)) = (0,1) = (1,1)(1,0) = \alpha((0,1))\alpha((1,1)) \tag{8}$$
$$\alpha((1,0)(0,1)) = \alpha((1,1)) = (1,0) = (0,1)(1,1) = \alpha((1,0))\alpha((0,1)) \tag{9}$$
$$\alpha((1,1)(1,1)) = \alpha((0,0)) = (0,0) = (1,0)(1,0) = \alpha((1,1))\alpha((1,1)) \tag{10}$$

Hence, α is an automorphism.
Similarly we find that every other permutation is also an automorphism. Hence, the automorphism group of $\mathbb{Z}_2^{\times 2}$ is S_3.

(b) In the lectures we saw that $\mathsf{Aut}\,(\mathbb{Z}_8) = \mathbb{Z}_8^{\star}$, the multiplicative group of integers modulo 8. The group elements are the integers coprime to 8. These are 1,3,5,7. The group multiplication is simply multiplication modulo 8. It is therefore straightforward to compute that $3^2 = 5^2 = 7^2 = 1 \mod 8$ and $7 = 3 \cdot 5 \mod 8$. This is exactly the group $\mathbb{Z}_2^{\times 2}$ that we already encountered above:

$$\mathbb{Z}_2^{\times 2} = \left\langle 3, 5 | 3^2 = 5^2 = 1, 3 \cdot 5 = 5 \cdot 3 \right\rangle. \tag{11}$$

Groups & Representations
Exercise session 3 - Solutions

April 1, 2022
16:00-17:30

1 Center subgroup, classes, quotient group

1.1. *The quaternion group* Q *is a group of order 8. Its group elements are* $\{\pm 1, \pm i, \pm j, \pm k\}$ *and the multiplication is given by* $i^2 = j^2 = k^2 = 1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. *It is clearly a non-abelian group as eg.* $ij \neq ji$.[1]

(a) *Compute the center* $Z(\mathsf{Q})$. *Verify it is a subgroup.*

(b) *Construct the conjugacy classes of* Q.

(c) *Argue that the order of all the elements in a given class is the same. Show that this is true for every group, not only* Q.

(d) *Show that* $\{1, -1\} \triangleleft \mathsf{Q}$ *and construct* $\mathsf{Q}/\{1, -1\}$.

(e) *Same exercise for* $\{\pm 1, \pm i\} \triangleleft \mathsf{Q}$.

Solution:

(a) We consider all eight group elements one by one and check which commute with all group elements.

Of course 1 commutes with every group element, and so does -1. No other group element commutes with the entire group. For example, i does not commute with j as $ij = k \neq -k = ji$.

Hence, $Z(\mathsf{Q}) \cong \mathbb{Z}_2$.

(b) From $h1h^{-1} = hh^{-1} = 1$ it follows that $c(1) = \{1\}$. Similarly $c(-1) = \{-1\}$.

Furthermore, we compute that $1i1 = i$, $ji(-j) = -jk = -i$ and $ki(-k) = kj = -i$, where we used that the inverse of j is $-j$ and the inverse of k is $-k$, hence $c(i) = \{i, -i\}$.

Similarly $c(j) = \{j, -j\}$ and $c(k) = \{k, -k\}$.

Notice that the conjugacy classes are nó subgroups (except of course for $c(1)$ which forms a trivial group).

(c) The order of 1 is of course 1, the order of -1 is 2, and the order of the other six group elements $\pm i, \pm j, \pm k$ are 4, for example $k^4 = (-1)^2 = 1$. Hence, there is one class, $c(1)$ with the only group element of order one. $c(-1)$ contains the only element of order 2. And all other classes all contain elements of order 4.

Now we show that for an arbitrary finite group G the elements within the same class all have the same order. Consider therefore an arbitrary group element $g \in \mathsf{G}$ and its corresponding class $c(g)$. If we denote the order of g by n, ie. n is the smallest

[1] Give an argument why $\mathsf{Q} \ncong \mathsf{D}_4$.

integer such that $g^n = e$, then we find that $(hgh^{-1})^n = hg^nh^{-1} = heh^{-1} = e$. Now suppose that there were an integer $k < n$ such that $(hgh^{-1})^k = hg^kh^{-1} = heh^{-1} = e$, then this would imply that k is an integer smaller than n such that $g^k = e$, which would be in contradiction with the fact that n is the smallest number with that property, hence n is the order of hgh^{-1}. We conclude that all elements in $c(g)$ have the same order. □

(d) Clearly $\{1, -1\}$ constitutes a $\mathbb{Z}_2$ subgroup of Q. Now we need to show it's a normal subgroup. We thus need to show that $hgh^{-1} \in \{1, -1\}$ for all $h \in \mathsf{Q}$ and $g = \pm 1$. This follows immediately from $h(\pm 1)h^{-1} = \pm 1$.

The group elements of $\mathsf{Q}/\{1, -1\}$ are exactly the cosets $h\{1, -1\}$ $h \in \mathsf{Q}$. Hence, $\mathsf{Q}/\{1, -1\} = \{\{1, -1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}\}$. Representatives for these cosets are $\pm 1, \pm i, \pm j$ and $\pm k$ respectively.

Now we work out the multiplication rules for $\mathsf{Q}/\{1, -1\}$. Recall that by definition $g_1\{1, -1\} \cdot g_2\{1, -1\} := (g_1g_2)\{1, -1\}$. From this definition it immediately follows that the Cayley table reads

	$\{1,-1\}$	$\{i,-i\}$	$\{j,-j\}$	$\{k,-k\}$
$\{1,-1\}$	$\{1,-1\}$	$\{i,-i\}$	$\{j,-j\}$	$\{k,-k\}$
$\{i,-i\}$	$\{i,-i\}$	$\{1,-1\}$	$\{k,-k\}$	$\{j,-j\}$
$\{j,-j\}$	$\{j,-j\}$	$\{k,-k\}$	$\{1,-1\}$	$\{i,-i\}$
$\{k,-k\}$	$\{k,-k\}$	$\{j,-j\}$	$\{i,-i\}$	$\{1,-1\}$

. (1)

If we make the identifications $\{1, -1\} \leftrightarrow (0, 0)$, $\{i, -i\} \leftrightarrow (1, 0)$, $\{j, -j\} \leftrightarrow (0, 1)$ and $\{k, -k\} \leftrightarrow (1, 1)$ we can conclude that $\mathsf{Q}/\{1, -1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(e) $\{\pm 1, \pm i\}$ is closed under multiplication and contains the identity, it thus forms a subgroup of Q. In particular, it forms a $\mathbb{Z}_4$ subgroup; indeed, this follows from the identification $1 \leftrightarrow 0, i \leftrightarrow 1, -1 = i^2 \leftrightarrow 2, -i = i^3 \leftrightarrow 3$ and the fact that $i^4 = 1$.

Let us check it is also a normal subgroup. From the previous exercise it follows that $g(\pm 1)g^{-1} \in \{\pm 1, \pm i\}$ for all $g \in \mathsf{Q}$. We thus only need to check that $g(\pm i)g^{-1} \in \{\pm 1, \pm i\}$ for all $g \in \mathsf{Q}$. For $g = \pm 1$ and $g = \pm i$ this is trivial. For $g = \pm k$ we find $(\pm k)(\pm i)(\mp k) = \mp kik = \pm kj = \mp i \in \{\pm 1, \pm i\}$ and for $g = \pm j$ we have $(\pm j)(\pm i)(\mp j) = \mp jij = \mp jk = \mp i \in \{\pm 1, \pm i\}$.

The cosets are $\{\pm 1, \pm i\}$ and $\{\pm j, \pm k\}$ with representatives $\pm 1, \pm i$ and $\pm j, \pm k$ respectively. The multiplication table we find reads

	$\{\pm 1,\pm i\}$	$\{\pm j,\pm k\}$
$\{\pm 1,\pm i\}$	$\{\pm 1,\pm i\}$	$\{\pm j,\pm k\}$
$\{\pm j,\pm k\}$	$\{\pm j,\pm k\}$	$\{\pm 1,\pm i\}$

. (2)

This is exactly the multiplication table of $\mathbb{Z}_2$. We conclude that $\mathsf{Q}/\{\pm 1, \pm i\} \cong \mathbb{Z}_2$. Of course this could be expected as the order of $\mathsf{Q}/\{\pm 1, \pm i\}$ is $8/4 = 2$ and $\mathbb{Z}_2$ is the only group of order 2.

2 Semidirect product

2.2. *(a) Construct* $\mathsf{Aut}(\mathbb{Z}_3)$.

(b) Construct all homomorphisms $\beta : \mathbb{Z}_2 \mapsto \mathsf{Aut}(\mathbb{Z}_3)$.

(c) *For all the β's you found, construct $\mathbb{Z}_3 \rtimes_\beta \mathbb{Z}_2$*

Solution:

(a) Recall that the Cayley table of $\mathbb{Z}_3$ reads

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{array}. \tag{3}$$

Recall from the previous exercise session that an automorphism has to map the identity to the identity (as it is an homomorphism). Hence, $\mathsf{Aut}(\mathbb{Z}_3)$ can at most contain two automorphisms: one that swaps the groups elements $1 \leftrightarrow 2$ and one that leaves all group elements unchanged, the identity. The latter is always an automorphism. So, we only need to check that the other candidate that we shall denote by φ and acts according to

$$\varphi(0) = 0 \tag{4a}$$
$$\varphi(1) = 2 \tag{4b}$$
$$\varphi(2) = 1 \tag{4c}$$

is indeed a homomorphism.
This is readily checked as

$$\varphi(1) + \varphi(1) = 2 + 2 = 1 = \varphi(1 + 1) \tag{5a}$$
$$\varphi(1) + \varphi(2) = \varphi(2) + \varphi(1) = 2 + 1 = 0 = \varphi(0) \tag{5b}$$
$$\varphi(2) + \varphi(2) = 1 + 1 = 2 = \varphi(2 + 2). \tag{5c}$$

Note that acting with φ twice is equivalent to acting with the identity automorphism and doing nothing. From this we conclude that $\mathsf{Aut}(\mathbb{Z}_3) \cong \mathbb{Z}_2 \cong S_2$. We can write the Cayley table of this automorphism group as

$$\begin{array}{c|cc} & \mathrm{id} & \varphi \\ \hline \mathrm{id} & \mathrm{id} & \varphi \\ \varphi & \varphi & \mathrm{id} \end{array}. \tag{6}$$

(b) To find all group homomorphisms $\beta : \mathbb{Z}_2 \mapsto \mathsf{Aut}(\mathbb{Z}_3)$ we again use the fact that β has to map the identity of $\mathbb{Z}_2$ to the identity automorphism in $\mathsf{Aut}(\mathbb{Z}_3) \cong \mathbb{Z}_2$. We then find two homomorphisms:

$$\beta^{(1)} \equiv \beta : 1 \mapsto \varphi \tag{7a}$$
$$\beta^{(2)} \equiv \mathbb{I} : 1 \mapsto \mathrm{id} \tag{7b}$$

(c) Let us first consider $\mathbb{I}$. Since $\mathbb{I}$ maps every element of $\mathbb{Z}_2$ to the identity automorphism, we have that $\beta_h(n) = n \ \forall h, n$. In other words, $(n_1, h_1) \cdot (n_2, h_2) := (n_1 n_2, h_1 h_2)$, which is by definition the multiplication rule of the direct product of $\mathbb{Z}_3 \times \mathbb{Z}_2$. In other words, $\mathbb{Z}_3 \rtimes_{\mathbb{I}} \mathbb{Z}_2 = \mathbb{Z}_3 \times \mathbb{Z}_2$.
Now we consider β. We have that $\beta_0(n) = n$ and $\beta_1(n) = -n$. We construct the multiplication table as follows

	(0,0)	(1,0)	(2,0)	(0,1)	(2,1)	(1,1)
(0,0)	(0,0)	(1,0)	(2,0)	(0,1)	(2,1)	(1,1)
(1,0)	(1,0)	(2,0)	(0,0)	(1,1)	(0,1)	(2,1)
(2,0)	(2,0)	(0,0)	(1,0)	(2,1)	(1,1)	(0,1)
(0,1)	(0,1)	(2,1)	(1,1)	(0,0)	(1,0)	(2,0)
(2,1)	(2,1)	(1,1)	(0,1)	(2,0)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(2,1)	(1,0)	(2,0)	(0,0)

. (8)

Clearly this group is not abelian, so it has to be isomorphic to D_3. Let us make this isomorphism explicit. If we write down the multiplication table of $\mathsf{D}_3 \cong \langle a, b | a^3 = b^2 = (ab)^2 = e \rangle$ (see previous exercise session) we obtain following Cayley table:

	e	a	a^2	b	ba	ba^2
e	e	a	a^2	b	ba	ba^2
a	a	a^2	e	ba^2	b	ba
a^2	a^2	e	a	ba	ba^2	b
b	b	ba	ba^2	e	a	a^2
ba	ba	ba^2	b	a^2	e	a
ba^2	ba^2	b	ba	a	a^2	e

. (9)

You can check this table is the same as (8) under the identification

$$(0,0) \leftrightarrow e \tag{10a}$$
$$(1,0) \leftrightarrow a \tag{10b}$$
$$(0,1) \leftrightarrow b. \tag{10c}$$

3 Representations

3.3. *Show that the Pauli matrices*

$$\sigma_X = \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_Y = \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_Z = \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

together with the 2×2 *identity matrix can be used to construct a faithful two-dimensional representation of* Q.

Solution: It is a straightforward to check that the Pauli matrices multiply according to

$$\sigma_a \sigma_b = \delta_{ab} \mathbb{1} + i \sum_c \varepsilon_{abc} \sigma_c. \tag{12}$$

From this it follows that

$$\sigma_{4-a} \sigma_{4-b} = \delta_{ab} \mathbb{1} - i \sum_c \varepsilon_{abc} \sigma_{4-c}. \tag{13}$$

The quaternions on the other hand multiply according to

$$i_a i_b = -\delta_{ab} + \sum_c \varepsilon_{abc} i_c, \qquad a, b, c = 1, 2, 3 \tag{14}$$

and where $i_1 = i, i_2 = j, i_3 = k$. Comparing the two, a faithful representation is found to be

$$\pm 1 \mapsto \pm \mathbb{1} \tag{15a}$$

$$\pm i \mapsto \pm i\sigma_X \tag{15b}$$

$$\pm j \mapsto \pm i\sigma_Y \tag{15c}$$

$$\pm k \mapsto \pm i\sigma_Z . \tag{15d}$$

Groups & Representations
Exercise session 4 - Solutions

April 22, 2022
16:00-17:30

1 Basic notions of the representation theory of finite groups

1.1. *(a) Show that all the irreducible representations of an abelian group are one-dimensional.*

(b) Construct all irreps of $\mathbb{Z}_k$. Check that there are as many irreps as there are conjugacy classes. Of particular importance are the irreps of $\mathbb{Z}_2$, write them down.

Solution:

(a) Suppose we denote a given *irreducible* representation by $\{X_g | g \in \mathsf{G}\}$. Since G is by assumption abelian, we have that $X_g X_h = X_h X_g = X_{gh}$ for all g and h. For fixed g, we thus have that X_g commutes with all the matrices X_h that make up the representation. By virtue of Schur's lemma, we thus find that $X_g = \lambda(g)\mathbb{1}$ for some complex number $\lambda(g)$ that may depend on the group element g. Since this is true for every g, we thus conclude that the representation is of the form $\{\lambda(g)\mathbb{1} | g \in \mathsf{G}\}$. However, since this representation is by assumption irreducible, its has to be one-dimensional, for if it were a higher-dimensional representation, this representation would reduce to multiple one-dimensional blocks.

(b) Recall that $\mathbb{Z}_k$ is generated by one element, call it g, that satisfies $g^k = e$, in presentation notation: $\mathbb{Z}_k = \langle g | g^k = e \rangle$. Since all irreps of $\mathbb{Z}_k$ are one-dimensional, we thus find that every irrep, let us denote them by α, is completely specified by a complex number corresponding to the generator g, $\lambda_g^{(\alpha)}$. This complex number satisfies $\left(\lambda_g^{(\alpha)}\right)^k = \lambda_{g^k}^{(\alpha)} = \lambda_e^{(\alpha)} = 1$. Hence, every irrep α corresponds to a root of unity: $\lambda_g^{(\alpha)} = \exp\left(\frac{2\pi i \alpha}{k}\right)$ where $\alpha = 0, 1, ..., k-1$. There are thus k one-dimensional irreps. There are also k conjugacy classes, as every conjugacy class of an abelian group contains only one element (see previous exercise session). Note that the irrep $\alpha = 0$ corresponds to the trivial representation that maps every group element to 1. For $\mathbb{Z}_2$, we find besides the trivial representation also the so-called *sign representation*: $\lambda_e^{(1)} = 1, \lambda_2^{(1)} = -1$, 2 being the non-trivial group element of $\mathbb{Z}_2$.

1.2. *(a) Write down all elements of S_3 as cycles.*

(b) How many classes does S_3 have? How many irreps do you expect?

(c) Construct the defining *representation of S_3. This representation is defined as $[D(p)]_{ij} = 1$ if $i = p(j)$ and 0 otherwise, $D(p) = \sum_j |p(j)\rangle\langle j|$. You do not have to check that this indeed forms a representation, but this can be a useful exercise to check.*

(d) *Consider following basis transformation:*

$$\tilde{\vec{e}}_1 = \frac{1}{\sqrt{3}}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \tag{1a}$$

$$\tilde{\vec{e}}_2 = \frac{1}{\sqrt{6}}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) \tag{1b}$$

$$\tilde{\vec{e}}_3 = \frac{1}{\sqrt{2}}(\vec{e}_2 - \vec{e}_3). \tag{1c}$$

Write down the basis transformation in matrix notation: $\tilde{\vec{e}}_i = \sum_j C_{ij}\vec{e}_j$. *Write down the inverse matrix* C^{-1}.

(e) *Apply the basis transformation* $CR(g)C^{-1}$ *to the defining representation. Which irreps do you find?*

(f) *Compute the characters of the irreps you found in the previous exercise. Make a character table of* S_3.

(g) *Construct the last missing irrep from the character table.*

(h) *Give a geometrical interpretation of the two-dimensional irrep.*

(i) *Which of these irreps are faithful?*

Solution:

(a) S_3 has six elements: the identity, three transpositions and two cyclic permutations. In cycle notation (we don't write 1-cycles):

$$(),\ (12),\ (23),\ (13),\ (123)\ \&\ (132). \tag{2}$$

(b) There are three classes corresponding to the identity[1], the transpositions and the cyclic permutations. We shall name them:

$$E = \{()\},\ 2C_3 = \{(123), (132)\}\ \&\ 3\sigma_V = \{(12), (23), (13)\}. \tag{3}$$

We expect S_3 to have three irreps as it has three conjugacy classes.

(c) It is obvious that $D(()) = \mathbb{1}_3$. A non-trivial example is $D((12))$, since (12) swaps $1 \leftrightarrow 2$, we find:

$$D((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Similarly:

$$D((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, D((13)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5}$$

And

$$D((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, D((132)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6}$$

[1] Recall that the identity always makes up its own conjugacy class.

(d) We find:

$$C = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{7}$$

we will also need

$$C^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{8}$$

(e) Reducing in irreps is then straightforward, but tedious. Writing $D'(p) = CD(p)C^{-1}$ it is found that

$$D(()) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D((12)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad D((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{9a}$$

$$D((13)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad D((123)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \tag{9b}$$

$$D((132)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{9c}$$

We note that every representation matrix is the diagonal sum of a one-dimensional block, corresponding to the trivial representation, and a two-dimensional block. Hence, we can write symbolically that

$$D = A_1 \oplus E, \tag{10}$$

where D, A_1 and E respectively denote the defining representation and the trivial - and two-dimensional irreducible representations.[2] This notation for these reps is customary in eg. molecular physics. The equality should be understood up to basis transformation.

(f) In the previous exercise we found two of three irreps of S_3. From the fact that (see theory lectures)

$$\sum_\alpha \left(d^{(\alpha)}\right)^2 = |\mathsf{G}| \tag{11}$$

we conclude that there is one one-dimensional irrep missing, we will denote it A_2. Let us complete the character table with the characters of A_2. Clearly, since A_2 is one-dimensional $\chi^{(A_2)}(E) = 1$. Furthermore we use the great orthogonality

[2] Non-examinable: that this two-dimensional block indeed constitutes an irrep and cannot further be reduced, follows for example from the fact that a rep V is irreducible if and only if $\frac{1}{|\mathsf{G}|}\sum_g \chi_V(g)\overline{\chi_V(g)} = 1$, which is indeed the case for this two-dimensional rep.

theorem applied to the characters (sometimes also called the *little orthogonality theorem*),

$$\frac{1}{|\mathsf{G}|}\sum_x \chi^{(\alpha)}(x)\bar{\chi}^{(\beta)}(x) = \delta_{\alpha\beta} \tag{12}$$

to write down following system of equation:

$$\begin{cases} 1 + 2\chi^{(A_2)}(2C_3) + 3\chi^{(A_2)}(3\sigma_V) & = 0 \\ 2 - 2\chi^{(A_2)}(2C_3) & = 0 \end{cases}, \tag{13}$$

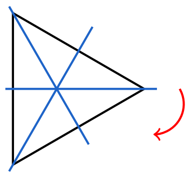
from which we immediately conclude that $\chi^{(A_2)}(2C_3) = 1$ and $\chi^{(A_2)}(3\sigma_V) = -1$. The character table thus reads

	E	$2C_3$	$3\sigma_V$
A_1	1	1	1
E	2	-1	0
A_2	1	1	-1

(14)

(g) Since this last irrep A_2 is one-dimensional, the representation matrices are just numbers and equal to the traces. This irrep thus maps every even permutation (the cyclic permutations) to 1 and every odd permutation (the transpositions) to -1. This irrep is also called the *sign representation* as it maps every permutation to its sign/parity.

(h) S_3 is isomorphic to the dihedral group of order 6, D_3. D_3 is the symmetry group of an equilateral triangle (see also exercise session 2, where we studied the example D_4). These symmetries are the identity, a (clockwise) rotation of 120°, a rotation of 240° and rotations of 180° around three principal axis:



Hence, D_3 is generated by a reflection around the horizontal blue axis (a) and a clockwise 120° rotation (b), prone to the constraints $a^2 = b^3 = (ab)^2 = e$, in presentation notation:

$$\mathsf{D}_3 = \langle a, b | a^2 = b^3 = (ab)^2 = e \rangle . \tag{15}$$

a can then be represented by the matrix

$$a \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{16}$$

whereas b can be represented by a rotation matrix

$$b \leftrightarrow \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \overset{\varphi=120}{=} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{17}$$

it's very easy to check that these matrices indeed satisfy the constraints mentioned above. These matrices exactly correspond to the two-dimensional representation matrices corresponding to the (23) and (312) elements of S_3. The other representation matrices can be found by taking products of a and b.

(i) The trivial representation maps every group element to 1 and is thus clearly not faithful. Similarly, the sign representation also maps multiple group elements to 1 and the other group elements to -1, also not injective. The two-dimensional representation on the other hand maps every group element to a distinct matrix and is thus faithful.

2 Tensor product of representations

2.3. *(a) Given a representation $\{U(g)|g \in \mathsf{G}\}$ of a (finite) group, show that the tensor product representation $\{U(g) \otimes U(g)|g \in \mathsf{G}\}$ also forms a representation of this group. What is its dimension?*

(b) What are the characters of the tensor product representation in function of the characters of the representation $\{U(g)|g \in \mathsf{G}\}$?

(c) Determine the dimension of $(V \otimes V)^S$ and $(V \otimes V)^A$.

(d) Show that $(V \otimes V)^S$ and $(V \otimes V)^A$ constitute subrepresentations of $V \otimes V$ and compute the characters in these subrepresentations.

Solution:

(a) This follows from $(U(g) \otimes U(g))(U(h) \otimes U(h)) = U(g)U(h) \otimes U(g)U(h) = U(gh) \otimes U(gh)$ where we used that the $U(g)$'s form a representation and the general rule

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}. \tag{18}$$

The dimension of the tensor product representation is d^2.

(b) $U(g) \otimes U(g)$ acts on the given tensor product basis according to

$$(U(g) \otimes U(g)) |\psi_{ij}\rangle = \sum_{i',j'} U(g)_{ii'} U(g)_{jj'} |\psi_{i'j'}\rangle \tag{19}$$

such that $[U(g) \otimes U(g)]_{ij,i'j'} = U(g)_{ii'} U(g)_{jj'}$[3], the character is then

$$\chi^{U \otimes U}(g) = \sum_{i,j} [U(g) \otimes U(g)]_{ij,ij} = U(g)_{ii} U(g)_{jj} = \left[\chi^U(g)\right]^2. \tag{20}$$

(c) $(V \otimes V)^A$ has $\binom{d}{2} = \frac{1}{2}d(d-1)$ different basis states $\left|\psi_{ij}^A\right\rangle$: there are d different choices for the label i, $d-1$ for j (as choosing $i = j$ would lead to a zero vector $\left|\psi_{ii}^A\right\rangle = 0$) and we have to divide by 2 as $\left|\psi_{ij}^A\right\rangle$ and $\left|\psi_{ji}^A\right\rangle$ are clearly linearly dependent.

Since $V \otimes V = (V \otimes V)^S \oplus (V \otimes V)^A$ and $\dim(V \otimes V) = d^2$, $\dim\left((V \otimes V)^A\right) = \frac{1}{2}d(d-1)$, we have that $\dim\left((V \otimes V)^S\right) = \frac{1}{2}d(d+1)$.

(d) We need to show $(U(g) \otimes U(g))(V \otimes V)^S \subset (V \otimes V)^S$ and similarly for $(V \otimes V)^A$.

[3] See also the section *Tensor product of linear maps* from the course on vector and function spaces.

Consider the action on the basis states:

$$\begin{aligned}
(U(g)\otimes U(g))\left|\psi^S_{ij}\right\rangle &= (U(g)\otimes U(g))\left|\psi_{ij}\right\rangle + (U(g)\otimes U(g))\left|\psi_{ji}\right\rangle && \text{(21a)}\\
&= \sum_{i',j'} U(g)_{ii'}U(g)_{jj'}\left|\psi_{i'j'}\right\rangle + U(g)_{ji'}U(g)_{ij'}\left|\psi_{i'j'}\right\rangle && \text{(21b)}\\
&= \sum_{i',j'} \left(U(g)_{ii'}U(g)_{jj'} + U(g)_{ji'}U(g)_{ij'}\right)\left|\psi_{i'j'}\right\rangle && \text{(21c)}\\
&= \frac{1}{2}\sum_{i',j'} \left(U(g)_{ii'}U(g)_{jj'} + U(g)_{ji'}U(g)_{ij'}\right)\left(\left|\psi^S_{i'j'}\right\rangle + \left|\psi^A_{i'j'}\right\rangle\right) && \text{(21d)}\\
&= \frac{1}{2}\sum_{i',j'} \left(U(g)_{ii'}U(g)_{jj'} + U(g)_{ji'}U(g)_{ij'}\right)\left|\psi^S_{i'j'}\right\rangle && \text{(21e)}\\
&=: \sum_{i',j'} (U(g)\otimes U(g))^S_{ij,i'j'}\left|\psi^S_{i'j'}\right\rangle . && \text{(21f)}
\end{aligned}$$

In the fourth line we wrote $\left|\psi_{i'j'}\right\rangle$ as $\left|\psi_{i'j'}\right\rangle = \frac{1}{2}\left(\left|\psi^S_{i'j'}\right\rangle + \left|\psi^A_{i'j'}\right\rangle\right)$ and in the fifth line we used the fact that the contraction of the symmetric (in i' and j') tensor between brackets and antisymmetric tensor $\left|\psi^A_{i'j'}\right\rangle$ is zero.
This demonstrates that the symmetric subspace is indeed a subrepresentation of the tensor product representation. This subrepresentation has characters

$$\chi^S(g) = \sum_{i,j} (U(g)\otimes U(g))^S_{ij,ij} \tag{22}$$

$$= \frac{1}{2}\sum_{i,j} U(g)_{ii}U(g)_{jj} + U(g)_{ij}U(g)_{ji} \tag{23}$$

$$= \frac{1}{2}\left(\left[\chi^U(g)\right]^2 + \chi^U(g^2)\right) . \tag{24}$$

Similarly for the antisymmetrized space:

$$\begin{aligned}
(U(g)\otimes U(g))\left|\psi^A_{ij}\right\rangle &= (U(g)\otimes U(g))\left|\psi_{ij}\right\rangle - (U(g)\otimes U(g))\left|\psi_{ji}\right\rangle && \text{(25a)}\\
&= \sum_{i',j'} U(g)_{ii'}U(g)_{jj'}\left|\psi_{i'j'}\right\rangle - U(g)_{ji'}U(g)_{ij'}\left|\psi_{i'j'}\right\rangle && \text{(25b)}\\
&= \sum_{i',j'} \left(U(g)_{ii'}U(g)_{jj'} - U(g)_{ji'}U(g)_{ij'}\right)\left|\psi_{i'j'}\right\rangle && \text{(25c)}\\
&= \frac{1}{2}\sum_{i',j'} \left(U(g)_{ii'}U(g)_{jj'} - U(g)_{ji'}U(g)_{ij'}\right)\left|\psi^A_{i'j'}\right\rangle && \text{(25d)}\\
&=: \sum_{i',j'} (U(g)\otimes U(g))^A_{ij,i'j'}\left|\psi^A_{i'j'}\right\rangle . && \text{(25e)}
\end{aligned}$$

and thus

$$\chi^S(g) = \sum_{i,j} (U(g)\otimes U(g))^S_{ij,ij} \tag{26}$$

$$= \frac{1}{2}\sum_{i,j} U(g)_{ii}U(g)_{jj} - U(g)_{ji}U(g)_{ij} \tag{27}$$

$$= \frac{1}{2}\left(\left[\chi^U(g)\right]^2 - \chi^U(g^2)\right) . \tag{28}$$

3 Projective representation

3.4. *The smallest group with a non-trivial projective representation is the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Check that a projective representation of this group is formed by the Pauli matrices:*

$$D((0,0)) = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; D((0,1)) = \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{29}$$

$$D((1,0)) = \sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; D((1,1)) = i\sigma_Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{30}$$

and find the 2-cocycles.

Solution:

This is simply an exercise in matrix multiplication. A useful identity that you can easily prove yourself is

$$\sigma_a \sigma_b = \delta_{ab} \mathbb{1} + i \sum_c \varepsilon_{abc} \sigma_c, \qquad a, b, c = X, Y, Z. \tag{31}$$

Hence, we find for example

$$D((1,0))D((0,1)) = \sigma_X \sigma_Z \tag{32a}$$
$$= -i\sigma_Y \tag{32b}$$
$$= -D((1,1)), \tag{32c}$$

from which we conclude that $\omega((1,0),(0,1)) = \pi$. Similarly we find that $\omega((1,0),(0,1)) = \omega((1,1),(0,1)) = \omega((1,0),(1,1)) = \pi$ all other ω's are 0.

Groups & Representations
Exercise session 5 - Solutions

April 29, 2022
16:00-17:30

1 Tensor product of representations

1.1. (a) *Given a representation* $\{U(g)|g \in \mathsf{G}\}$ *of a (finite) group* G*, show that the tensor product* $\{U(g) \otimes U(g)|g \in \mathsf{G}\}$ *also forms a representation of this group. What is its dimension?*

(b) *What are the characters of the tensor product representation in function of the characters of the representation* $\{U(g)|g \in \mathsf{G}\}$*?*

Solution:

(a) This follows from $(U(g) \otimes U(g))(U(h) \otimes U(h)) = U(g)U(h) \otimes U(g)U(h) = U(gh) \otimes U(gh)$ where we used that the $U(g)$'s form a representation and the general rule

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}. \tag{1}$$

The dimension of the tensor product representation is d^2.

(b) $U(g) \otimes U(g)$ acts on the given tensor product basis according to

$$(U(g) \otimes U(g)) |\psi_{ij}\rangle = \sum_{i',j'} U(g)_{ii'} U(g)_{jj'} |\psi_{i'j'}\rangle \tag{2}$$

such that $[U(g) \otimes U(g)]_{ij,i'j'} = U(g)_{ii'} U(g)_{jj'}$[1], the character is then

$$\chi^{U \otimes U}(g) = \sum_{i,j} [U(g) \otimes U(g)]_{ij,ij} = \sum_i U(g)_{ii} \sum_j U(g)_{jj} = \left[\chi^U(g)\right]^2. \tag{3}$$

2 Characters of group representations

2.2. *Verify that the chararacters of the irreps of* $\mathbb{Z}_N$ *are orthonormal. Hint: you can use* $\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}(n-n')k} = \delta_{n,n'}$.

Solution:

Recall that the irreps of $\mathbb{Z}_N = \langle g | g^N = e \rangle$ are given by $\lambda^{(\alpha)}_{g^k} = \exp\left(\frac{2\pi i \alpha k}{N}\right)$, $\alpha = 0, 1, ..., N-1$. Hence,

$$\left\langle \chi^{(\alpha)} \middle| \chi^{(\beta)} \right\rangle = \frac{1}{N} \sum_k e^{\frac{-2\pi i \alpha k}{N}} e^{\frac{2\pi i \beta k}{N}} \tag{4a}$$

$$= \frac{1}{N} \sum_k e^{\frac{2\pi i (\beta - \alpha) k}{N}} \tag{4b}$$

$$= \delta_{\alpha,\beta}. \tag{4c}$$

[1] See also the section *Tensor product of linear maps* from the course on vector and function spaces.

Where in the first line we used $|\mathbb{Z}_N| = N$ and carried out the sum over the group elements, and in the second line we used the identity from the hint.

2.3. *Recall the character table of S_3 from a previous exercise:*

$$\begin{array}{c|ccc} & E & 2C_3 & 3\sigma_V \\ \hline A_1 & 1 & 1 & 1 \\ E & 2 & -1 & 0 \\ A_2 & 1 & 1 & -1 \end{array} . \tag{5}$$

(a) *Compute the character of $E \otimes E$ and decompose in irreps.*

(b) *Compute the character of $E \otimes E \otimes E$ and decompose in irreps.*

(c) *Compute the character of $E^{\otimes N}$, decompose in irreps and check for $N = 2, 3$ that you reobtain the previous results.*

Solution:

(a) We have that $\chi^{(E\otimes E)} = (2^2, (-1)^2, 0^2) = (4, 1, 0)$, which can be written as $\chi^{(E\otimes E)} = \chi^{A_1} + \chi^{A_2} + \chi^{E}$, hence: $E \otimes E \cong A_1 \oplus A_2 \oplus E$.

(b) Similarly, we find $\chi^{(E\otimes E\otimes E)} = (8, -1, 0)$. Let us compute following inner products:

- $\langle \chi^{(E\otimes E\otimes E)} | \chi^{(A_1)} \rangle = \frac{1}{6}(1 \cdot 8 \cdot 1 + 2 \cdot (-1) \cdot 1 + 3 \cdot 0 \cdot 1) = 1$
- $\langle \chi^{(E\otimes E\otimes E)} | \chi^{(E)} \rangle = \frac{1}{6}(1 \cdot 8 \cdot 2 + 2 \cdot (-1) \cdot (-1) + 3 \cdot 0 \cdot 0) = 3$
- $\langle \chi^{(E\otimes E\otimes E)} | \chi^{(A_2)} \rangle = \frac{1}{6}(1 \cdot 8 \cdot 1 + 2 \cdot (-1) \cdot 1 + 3 \cdot 0 \cdot (-1)) = 1$,

and thus: $E \otimes E \otimes E \cong A_1 \oplus E \oplus E \oplus E \oplus A_2$.

(c) In general: $\chi^{(E^{\otimes N})} = (2^N, (-1)^N, 0)$,

- $\left\langle \chi^{(E^{\otimes N})} \middle| \chi^{(A_1)} \right\rangle = \frac{1}{6}\left(1 \cdot 2^N \cdot 1 + 2 \cdot (-1)^N \cdot 1 + 3 \cdot 0 \cdot 1\right) = \frac{1}{6}\left(2^N + 2 \cdot (-1)^N\right)$
- $\left\langle \chi^{(E^{\otimes N})} \middle| \chi^{(E)} \right\rangle = \frac{1}{6}\left(1 \cdot 2^N \cdot 2 + 2 \cdot (-1)^N \cdot (-1) + 3 \cdot 0 \cdot 0\right) = \frac{1}{6}\left(2^{N+1} + 2 \cdot (-1)^{N+1}\right)$
- $\left\langle \chi^{(E^{\otimes N})} \middle| \chi^{(A_2)} \right\rangle = \frac{1}{6}\left(1 \cdot 2^N \cdot 1 + 2 \cdot (-1)^N \cdot 1 + 3 \cdot 0 \cdot (-1)\right) = \frac{1}{6}\left(2^N + 2 \cdot (-1)^N\right)$,

from which we indeed find the previous results for $N = 2, 3$.

2.4. *Recall the quaternion group Q from a previous exercise session: its 8 group elements are $\{\pm 1, \pm i, \pm j, \pm k\}$ and the multiplication is given by $i^2 = j^2 = k^2 = 1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. A faithful two-dimensional representation was given by the Pauli matrices:*

$$\pm 1 \mapsto \pm \mathbb{1} \tag{6a}$$
$$\pm i \mapsto \pm i\sigma_X \tag{6b}$$
$$\pm j \mapsto \pm i\sigma_Y \tag{6c}$$
$$\pm k \mapsto \pm i\sigma_Z. \tag{6d}$$

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

Let us denote this representation by P.

(a) *From the relation*

$$\sum_{\alpha} \left(d^{(\alpha)}\right)^2 = |\mathsf{G}| \tag{8}$$

find out how many irreps there are and what their corresponding dimensions are.

(b) *Construct a character table and fill in the characters of P.*

(c) *Explicitly find all one-dimensional irreps. Hint: begin by showing that all the characters of the one-dimensional irreps are ± 1.*

(d) *Decompose $P \otimes P$.*

Solution:

(a) $|\mathsf{Q}| = 8$, we have already found one two-dimensional representation and there is always the trivial one-dimensional representation. Hence, we find that the dimensions of the missing representations squared are equal to three. This is only possible if there are three missing irreps of dimension one.

(b) Recalling the conjugacy classes we found in a previous exercise, we already have that

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
E	1	1	1	1	1
R_1	1	?	?	?	?
R_2	1	?	?	?	?
R_3	1	?	?	?	?
P	2	-2	0	0	0

(9)

where the R_i denote the (non-trivial) one-dimensional irreps and E is the trivial irrep.

(c) First note that the characters of the unknown one-dimensional irreps are exactly equal to the irreps themselves.

Also notice that all the ?'s missing in the table will correspond to either $+1$ or -1 as eg. $(-1)^2 = 1$ and thus $\left(\chi^{(R_{1,2,3})}(-1)\right)^2 = \chi^{(R_{1,2,3})}(1) = 1$ and eg. $i \cdot (-i) = 1 = \chi^{(R_{1,2,3})}(i)\chi^{(R_{1,2,3})}(-i) = \left(\chi^{(R_{1,2,3})}(i)\right)^2$.

Furthermore, we know that $R_{1,2,3}$ has to map the group element -1 to the "matrix" 1 because if it were -1, then we would have eg. that $\chi^{(R_{1,2,3})}(i)\chi^{(R_{1,2,3})}(-1) = -\chi^{(R_{1,2,3})}(-i) = -\chi^{(R_{1,2,3})}(i)$ and thus that $\chi^{(R_{1,2,3})}(i) = 0$, in conflict with the fact that $\chi^{(R_{1,2,3})}(i) = \pm 1$. So far we thus have

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
E	1	1	1	1	1
R_1	1	1	± 1	± 1	± 1
R_2	1	1	± 1	± 1	± 1
R_3	1	1	± 1	± 1	± 1
P	2	-2	0	0	0

(10)

Consider now some fixed R_i. From $ij = k$, it follows that $\chi^{(R_{1,2,3})}(i)\chi^{(R_{1,2,3})}(j) = \chi^{(R_{1,2,3})}(k)$. In other words, if we know $\chi^{(R_{1,2,3})}(i)$ and $\chi^{(R_{1,2,3})}(j)$, we know $\chi^{(R_{1,2,3})}(k)$. Given these constraints, we immediately conclude that the full character table is given by

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
E	1	1	1	1	1
R_1	1	1	1	-1	-1
R_2	1	1	-1	1	-1
R_3	1	1	-1	-1	1
P	2	-2	0	0	0

(11)

(d) $\chi^{P\otimes P} = (4, 4, 0, 0, 0)$, which can be written as a sum of the characters of the four one-dimensional irreps, hence $P \otimes P = E \oplus R_1 \oplus R_2 \oplus R_3$.[2]

[2] As extra exercise: diagonalize $i\sigma_X \otimes i\sigma_X$ and show that its eigenbasis is the basis in which $P \otimes P$ reduces in the four aforementioned one-dimensional irreps.

Groups & Representations
Exercise session 6 - solutions

May 6, 2022
16:00-17:30

1 Characters of group representations

1.1. *Recall the quaternion group* Q *from a previous exercise session: its 8 group elements are* $\{\pm 1, \pm i, \pm j, \pm k\}$ *and the multiplication is given by* $i^2 = j^2 = k^2 = 1$*,* $ij = -ji = k$*,* $jk = -kj = i$*,* $ki = -ik = j$*. A faithful two-dimensional representation was given by the Pauli matrices:*

$$\pm 1 \mapsto \pm \mathbb{1} \tag{1a}$$
$$\pm i \mapsto \pm i\sigma_X \tag{1b}$$
$$\pm j \mapsto \pm i\sigma_Y \tag{1c}$$
$$\pm k \mapsto \pm i\sigma_Z. \tag{1d}$$

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Let us denote this representation by P*.*

(a) *From the relation*

$$\sum_\alpha \left(d^{(\alpha)}\right)^2 = |\mathsf{G}| \tag{3}$$

find out how many irreps there are and what their corresponding dimensions are.

(b) *Construct a character table and fill in the characters of* P*.*

(c) *Explicitly find all one-dimensional irreps. Hint: begin by showing that all the characters of the one-dimensional irreps are* ± 1*.*

(d) *Decompose* $P \otimes P$*.*

Solution:

(a) $|\mathsf{Q}| = 8$, we have already found one two-dimensional representation and there is always the trivial one-dimensional representation. Hence, we find that the dimensions of the missing representations squared are equal to three. This is only possible if there are three missing irreps of dimension one.

(b) Recalling the conjugacy classes we found in a previous exercise, we already have that

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
E	1	1	1	1	1
R_1	1	?	?	?	?
R_2	1	?	?	?	?
R_3	1	?	?	?	?
P	2	-2	0	0	0

(4)

where the R_i denote the (non-trivial) one-dimensional irreps and E is the trivial irrep.

(c) First note that the characters of the unknown one-dimensional irreps are exactly equal to the irreps themselves.

Also notice that all the ?'s missing in the table will correspond to either $+1$ or -1 as eg. $(-1)^2 = 1$ and thus $\left(\chi^{(R_{1,2,3})}(-1)\right)^2 = \chi^{(R_{1,2,3})}(1) = 1$ and eg. $i \cdot (-i) = 1 = \chi^{(R_{1,2,3})}(i)\chi^{(R_{1,2,3})}(-i) = \left(\chi^{(R_{1,2,3})}(i)\right)^2$.

Furthermore, we know that $R_{1,2,3}$ has to map the group element -1 to the "matrix" 1 because if it were -1, then we would have eg. that $\chi^{(R_{1,2,3})}(i)\chi^{(R_{1,2,3})}(-1) = -\chi^{(R_{1,2,3})}(-i) = -\chi^{(R_{1,2,3})}(i)$ and thus that $\chi^{(R_{1,2,3})}(i) = 0$, in conflict with the fact that $\chi^{(R_{1,2,3})}(i) = \pm 1$. So far we thus have

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
E	1	1	1	1	1
R_1	1	1	± 1	± 1	± 1
R_2	1	1	± 1	± 1	± 1
R_3	1	1	± 1	± 1	± 1
P	2	-2	0	0	0

(5)

Consider now some fixed R_i. From $ij = k$, it follows that $\chi^{(R_{1,2,3})}(i)\chi^{(R_{1,2,3})}(j) = \chi^{(R_{1,2,3})}(k)$. In other words, if we know $\chi^{(R_{1,2,3})}(i)$ and $\chi^{(R_{1,2,3})}(j)$, we know $\chi^{(R_{1,2,3})}(k)$. Given these constraints, we immediately conclude that the full character table is given by

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
E	1	1	1	1	1
R_1	1	1	1	-1	-1
R_2	1	1	-1	1	-1
R_3	1	1	-1	-1	1
P	2	-2	0	0	0

(6)

(d) $\chi^{P\otimes P} = (4, 4, 0, 0, 0)$, which can be written as a sum of the characters of the four one-dimensional irreps, hence $P \otimes P = E \oplus R_1 \oplus R_2 \oplus R_3$.[1]

2 SU_2 and its Lie algebra $\mathfrak{su}(2)$

2.2. *(a) Show that* $\mathsf{SU}_N := \left\{M \in \mathbb{C}^{N\times N} \,|\, \det M = 1, M^\dagger = M^{-1}\right\}$ *forms a group. It is called the* special unitary group. *'Special' refers to the determinant being* 1, *'unitary' because all matrices are unitary* ($M^\dagger = M^{-1}$)*.*

(b) Consider the exponential map that allows us to write every element $M \in \mathsf{SU}_N$ close to the identity *as an exponential of its Lie algebra* $\mathfrak{su}(N)$*:*

$$M = \exp\left(i \sum_a \theta_a T^a\right), \qquad \theta_a \in \mathbb{R}. \tag{7}$$

Impose $\det M = 1$ *and* $M^\dagger = M^{-1}$ *to check which matrices* $\mathfrak{su}(N)$ *contains. What is the dimension of* $\mathfrak{su}(N)$*? Hint: use* $\det \exp = \exp \operatorname{tr}$*.*

Solution:

[1] As extra exercise: diagonalize $i\sigma_X \otimes i\sigma_X$ and show that its eigenbasis is the basis in which $P \otimes P$ reduces in the four aforementioned one-dimensional irreps.

(a) If $M_1, M_2 \in \mathsf{SU}_N$, then clearly $\det M_1 M_2 = \det M_1 \det M_2 = 1$ and also $(M_1 M_2)^\dagger = M_2^\dagger M_1^\dagger = M_2^{-1} M_1^{-1} = (M_1 M_2)^{-1}$. So the multiplication in SU_n is closed. Furthermore, it is associative (as the matrix multiplication is always associative) and $\mathbb{1}_N \in \mathsf{SU}_N$ is the unit for the multiplication in SU_N. We also notice that if $M \in \mathsf{SU}_N$, then also $M^{-1} \in \mathsf{SU}_N$. Hence, all the group axioms are satisfied.

(b) From $M^{-1} = \exp(-i \sum_a \theta_a T^a)$ and $M^\dagger = \exp\left(-i \sum_a \theta_a T^{a\dagger}\right)$ we conclude that the generators $\{T^a\}$ are hermitian and from $\det \exp(i \sum_a \theta_a T^a) = \exp(i \sum_a \theta_a \operatorname{Tr} T^a) \overset{!}{=} 1$, which has to be true for all choices of the parameters θ_a, it follows that the generators are also traceless. In other words: $\mathfrak{su}(N)$ is the *real* vector space of hermitian and traceless matrices. Notice that these matrices themselves don't have to be real! A general $N \times N$ complex matrix has $2N^2$ real parameters. There are $N(N-1)/2$ independent complex or thus $N(N-1)$ real off-diagonal parameters. The diagonal entries of a hermitian matrix are real, of which there are N, the tracelessness condition fixes one of those, leaving us with $\dim \mathfrak{su}(N) = N^2 - 1$.

2.3. Consider SU_2.

(a) Consider a general 2×2 matrix

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \tag{8}$$

Impose $\det M = 1$ and $M^\dagger = M^{-1}$, conclude that M can be written as

$$M = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{9}$$

Count the number of free real parameters M has. Is this in accordance with what you found in the previous exercise?

(b) Show that every element of SU_2 can be written as

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = a\mathbb{1}_2 + ib\sigma_X + ic\sigma_Y + id\sigma_Z, \qquad a, b, c, d \in \mathbb{R}. \tag{10}$$

Find the parameters a, b, c, d in function of α and β.

(c) We denote $t_i := \frac{\sigma_i}{2}$ and $\sigma_{\vec{n}} := n_x t_x + n_y t_y + n_z t_z$, $\vec{n}$ being a real unit vector. Show that $(\sigma_{\vec{n}})^2 = \frac{\mathbb{1}_2}{4}$.

(d) Show that the SU_2 matrices can also be parametrized as

$$R_{\vec{n}}(\theta) = e^{i\theta\sigma_{\vec{n}}} = \cos\left(\frac{\theta}{2}\right)\mathbb{1}_2 + 2i\sin\left(\frac{\theta}{2}\right)\sigma_{\vec{n}}, \qquad \theta \in [0, 2\pi[\tag{11}$$

by writing $e^{i\theta\sigma_{\vec{n}}}$ as a Taylor expansion.

(e) Expand $R_{\vec{n}}(\delta\theta)$ to first order in $\delta\theta$, find the generators of the Lie algebra $\mathfrak{su}(2)$ and compute its commutation relations.

Solution:

(a) $\det M = 1$ is equivalent to $\alpha\delta - \beta\gamma = 1$ and $M^\dagger = M^{-1}$ leads to

$$\begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{pmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \tag{12a}$$

$$= \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}. \tag{12b}$$

From which we can conclude that $\beta = -\overline{\gamma}$ and $\delta = \overline{\alpha}$, and thus

$$M = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{13}$$

α and β each parameterize two real parameters and the condition on the determinant eleminates one of these degrees of freedom, leaving us with $2+2-1=3$ real parameters, in accordance with the previous result.

(b) Choosing $\alpha = a + id,\ \beta = -c + ib$ results in

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} a+id & c+ib \\ -c+ib & a-id \end{pmatrix} = a\mathbb{1}_2 + ib\sigma_X + ic\sigma_Y + id\sigma_Z. \tag{14}$$

(c) We compute:

$$\begin{aligned} (\sigma_{\vec{n}})^2 &= n_x^2 \frac{\sigma_X^2}{4} + n_y^2 \frac{\sigma_Y^2}{4} + n_z^2 \frac{\sigma_z^2}{4} + \frac{n_x n_y}{4} (\sigma_X \sigma_Y + \sigma_Y \sigma_X) \\ &\quad + \frac{n_x n_z}{4} (\sigma_X \sigma_Z + \sigma_Z \sigma_X) + \frac{n_y n_z}{4} (\sigma_Y \sigma_Z + \sigma_Z \sigma_Y) && \text{(15a)} \\ &= \left(n_x^2 + n_y^2 + n_z^2\right) \frac{\mathbb{1}_2}{4} && \text{(15b)} \\ &= \frac{\mathbb{1}_2}{4}. && \text{(15c)} \end{aligned}$$

We used that $\{\sigma_i, \sigma_j\} := \sigma_i \sigma_j + \sigma_j \sigma_i = \delta_{i,j} \mathbb{1}_2$ and the fact that $\vec{n}$ is a unit vector.

(d)

$$\begin{aligned} R_{\vec{n}}(\theta) &= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} (\sigma_{\vec{n}})^k && \text{(16a)} \\ &= \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} (\sigma_{\vec{n}})^{2k} + \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} (\sigma_{\vec{n}})^{2k+1} && \text{(16b)} \\ &= \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} \frac{\mathbb{1}_2}{2^{2k}} + 2\sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} \frac{\sigma_{\vec{n}}}{2^{2k+1}} && \text{(16c)} \\ &= \sum_{k=0}^{\infty} (-)^k \frac{\theta^{2k}}{(2k)!} \frac{\mathbb{1}_2}{2^{2k}} + 2i \sum_{k=0}^{\infty} (-)^k \frac{\theta^{2k+1}}{(2k+1)!} \frac{\sigma_{\vec{n}}}{2^{2k+1}} && \text{(16d)} \\ &= \cos\left(\frac{\theta}{2}\right) \mathbb{1}_2 + 2i \sin\left(\frac{\theta}{2}\right) \sigma_{\vec{n}} && \text{(16e)} \end{aligned}$$

Where we used the results from (2.3c).

(e)

$$R_{\vec{n}}(\delta\theta) = \mathbb{1}_2 + i\delta\theta\sigma_{\vec{n}}, \tag{17}$$

where

$$\delta\theta\sigma_{\vec{n}} = \delta\theta n_x t_x + \delta\theta n_y t_y + \delta\theta n_z t_z. \tag{18}$$

From this we conclude that the Lie algebra of SU_2, $\mathfrak{su}(2)$, has three generators: t_x, t_y, t_z. The Lie algebra $\mathfrak{su}(2)$ can thus be written as

$$\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}} \{t_x, t_y, t_z\} \tag{19}$$

Using

$$[\sigma_i, \sigma_j] = 2i \sum_k \varepsilon_{ijk} \sigma_k, \tag{20}$$

it follows that the Lie bracket is given by

$$[t_i, t_j] = \sum_k i\varepsilon_{ijk} t_k. \tag{21}$$

The structure constants are thus $f_{ij}^k = i\varepsilon_{ijk}$. [2]

[2]Notice that $\mathfrak{su}(2)$ is a real vector space, even though the structure constants are imaginary.

Groups & Representations
Exercise session 7 - solutions

May 13, 2022
14:30-17:30

1 Representations of Lie algebras

1.1. *In this exercise we construct the so called* adjoint representation *of a Lie algebra* $\mathfrak{g}$. *It is defined as*

$$R_{ad} : \mathfrak{g} \mapsto \mathsf{GL}(\mathfrak{g}), \qquad x \mapsto ad_x, \tag{1}$$

where

$$ad_x(y) = [x, y]. \tag{2}$$

(a) Show that

$$([ad_x, ad_y])(z) = ad_{[x,y]}(z), \qquad \forall z \in \mathfrak{g}, \tag{3}$$

which demonstrates that R_{ad} is a Lie algebra homomorphism.

(b) Given that

$$\left[T^a, T^b\right] = \sum_c f_c^{ab} T^c, \tag{4}$$

show that

$$\left(R_{ad}\left(T^a\right)\right)^b{}_c = -f_c^{ab}. \tag{5}$$

(c) What will be the dimension of the adjoint representation of $\mathfrak{su}(2)$? Construct this representation.

Solution:

(a) From the definition of the adjoint representation it immediately follows that

$$([\mathrm{ad}_x, \mathrm{ad}_y])(z) = \mathrm{ad}_x\left(\mathrm{ad}_y(z)\right) - \mathrm{ad}_y\left(\mathrm{ad}_x(z)\right) =: [x, [y, z]] - [y, [x, z]]. \tag{6}$$

Application of the anti-symmetry of the Lie bracket and the Jacobi identity allows us to write this as

$$[x, [y, z]] - [y, [x, z]] = [x, [y, z]] + [y, [z, x]] = -[z, [x, y]] = [[x, y], z] := \mathrm{ad}_{[x,y]}(z) \tag{7}$$

(b) By definition:

$$\mathrm{R_{ad}}\left(T^a\right) = \left[T^a, \cdot\,\right]. \tag{8}$$

In the basis $\{T^i | i = 1, 2, ..., \dim \mathfrak{g}\}$, the adjoint thus acts on these basis vectors according to

$$\left(\mathrm{R_{ad}}\left(T^a\right)\right) T^c = \left[T^a, T^c\right] \tag{9a}$$

$$= \sum_c f_b^{ac} T^b, \tag{9b}$$

and thus:

$$\left(\mathrm{R}_{\mathrm{ad}}\left(T^a\right)\right)^b{}_c = -f^{ab}_c. \tag{10}$$

Note the minus sign that arises because the structure factors are antisymmetric in all indices, hence $f^{ab}_c = -f^{ac}_b$.

(c) The dimension of the adjoint representation is the dimension of the Lie algebra, hence the adjoint representation of $\mathfrak{su}(2)$ is three-dimensional. Using $f^{ab}_c = i\varepsilon_{abc}$, we find:

$$\mathrm{R}_{\mathrm{ad}}(t_x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{11a}$$

$$\mathrm{R}_{\mathrm{ad}}(t_y) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{11b}$$

$$\mathrm{R}_{\mathrm{ad}}(t_z) = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11c}$$

2 SO_3 and its Lie algebra $\mathfrak{so}(3)$

2.2. *Consider the exponential map that allows us to write every element* $M \in \mathsf{SO}_N$ close to the identity *as an exponential of its Lie algebra* $\mathfrak{so}(N)$*:*

$$M = \exp\left(i \sum_a \theta_a T^a\right), \qquad \theta_a \in \mathbb{R}. \tag{12}$$

Impose $\overline{M} = M$[1]*,* $\det M = 1$ *and* $M^\top = M^{-1}$ *to check which matrices* $\mathfrak{so}(N)$ *contains. What is the dimension of* $\mathfrak{so}(N)$*? Hint: use* $\det\exp = \exp\mathrm{tr}$*.*

Solution: First impose that M is real: $\overline{M} = \exp\bigl(-i\sum_a \theta_a \overline{T^a}\bigr) \stackrel{!}{=} M = \exp(i\sum_a \theta_a T^a)$, since this has to be true for all parameters, we find that the generators are purely imaginary. As in the case of SU_2 we find that the generators are traceless. Imposing $M^\top = M^{-1}$ gives $M^\top = \exp\bigl(i\sum_a \theta_a (T^a)^\top\bigr) \stackrel{!}{=} M^{-1} = \exp(-i\sum_a \theta_a T^a)$, the generators are thus antisymmetric. In other words, $\mathfrak{so}(N)$ is the *real* Lie algebra spanned by imaginary antisymmetric matrices, notice that antisymmetric matrices are always traceless. There are $\frac{N(N-1)}{2}$ generators, because there are $\binom{N}{2} = \frac{N(N-1)}{2}$ independent off-diagonal matrix elements which each have one real degree of freedom (as all entries have to be purely imaginary). Hence, $\dim_{\mathbb{R}}(\mathfrak{so}(N)) = \frac{N(N-1)}{2}$.

The dimension also makes sense from a geometrical point of view as SO_N describes the rotations in N spatial dimensions and there are $\frac{N(N-1)}{2}$ fundamental rotations, corresponding to the rotations in $\frac{N(N-1)}{2}$ planes (in case of SO_3 rotations in the xy -, yz -, xz-planes).

[1] Notice we did not have to do this in case of SU_2 as matrices in SU_2 are in general complex.

2.3. *Consider now* SO_3. *Every group element of* SO_3 *can be written as the product of three fundamental rotations, for example one around the* x, y *and* z *axis:*

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \tag{13a}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{13b}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13c}$$

(a) *Expand each of these matrices to first order in* $\delta\theta$, $R(\delta\theta) = \mathbb{1}_3 + i\delta\theta J_i$ *and read of the generators.*

(b) *Find the structure factors of* $\mathfrak{so}(3)$ *and show that* $\mathfrak{so}(3) \cong \mathfrak{su}(2)$.

(c) *Show that following operators* realise $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, *ie. satisfy the commutation relations of* $\mathfrak{so}(3) \cong \mathfrak{su}(2)$:

$$\widehat{J}_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \tag{14a}$$

$$\widehat{J}_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \tag{14b}$$

$$\widehat{J}_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{14c}$$

Notice that these operators correspond to the components of the quantum mechanical operator $\widehat{\vec{L}} = -\widehat{\vec{r}} \times \vec{\nabla}$. *Which can be derived from the classical angular momentum vector* $\vec{L} = \vec{r} \times \vec{p}$ *by substituting* $\vec{p} \mapsto -i\vec{\nabla}$.[2]

Solution:

(a) We find for $R_x(\delta\theta)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta\theta & \sin\delta\theta \\ 0 & -\sin\delta\theta & \cos\delta\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\theta \\ 0 & -\delta\theta & 1 \end{pmatrix} = \mathbb{1}_3 + i\delta\theta \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}}_{J_x}, \tag{15}$$

similarly we find:

$$J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tag{16a}$$

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{16b}$$

As expected we find three antisymmetric imaginary matrices that span $\mathfrak{so}(3) = \mathrm{span}_{\mathbb{R}}\{J_x, J_y, J_z\}$.

[2] We take $\hbar = 1$.

(b) There are two ways we can find the structure factors. One way is to compute the commutators $[J_i, J_j]$ by hand. However, we can also immediately conclude from (11a-11c) that the generators of $\mathfrak{so}(3)$ are the same as the (faithful) adjoint representation of $\mathfrak{su}(2)$, hence the structure constant of $\mathfrak{so}(3)$ are also $f_c^{ab} = i\varepsilon_{abc}$. This also immediately shows that $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, where the isomorphism is given by $J_x \leftrightarrow t_x, J_y \leftrightarrow t_y, J_z \leftrightarrow t_z$.

The generators of a Lie algebra form what we call the *defining representation* of that Lie algebra. In case of $\mathfrak{su}(2)$ formed by the Pauli matrices. What we have shown is thus that the defining representation of $\mathfrak{so}(3)$ is equivalent to the adjoint representation of $\mathfrak{su}(2)$. It is clear that the adjoint representation $\mathfrak{so}(3)$ will also be equivalent to the defining representation of $\mathfrak{so}(3)$ and the adjoint representation of $\mathfrak{su}(2)$.

(c) We will show this for $\widehat{J}_x$ and $\widehat{J}_y$ the other computations are similar. To compute its commutator, we need to let it act on a function $\psi \equiv \psi(x, y, z)$:

$$
\begin{aligned}
[\widehat{J}_x, \widehat{J}_y]\psi =& (-i)^2 \left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \right] \psi \\
=& - \left(y\frac{\partial}{\partial z} \left(z\frac{\partial}{\partial x}\psi \right) - z\frac{\partial}{\partial y} \left(z\frac{\partial}{\partial x}\psi \right) - y\frac{\partial}{\partial z} \left(x\frac{\partial}{\partial z}\psi \right) + z\frac{\partial}{\partial y} \left(x\frac{\partial}{\partial z}\psi \right) \right. \\
& \left. - z\frac{\partial}{\partial x} \left(y\frac{\partial}{\partial z}\psi \right) + x\frac{\partial}{\partial z} \left(y\frac{\partial}{\partial z}\psi \right) + z\frac{\partial}{\partial x} \left(z\frac{\partial}{\partial y}\psi \right) - x\frac{\partial}{\partial z} \left(z\frac{\partial}{\partial y}\psi \right) \right) \\
=& - \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right) \psi \\
=& i\widehat{J}_z\psi,
\end{aligned}
$$

such that $[\widehat{J}_x, \widehat{J}_y] = i\widehat{J}_z$, and more generally you can check that $[\widehat{J}_a, \widehat{J}_b] = i\sum_c \varepsilon_{abc}\widehat{J}_c$

3 The 2-to-1 homomorphism $\mathsf{SU}_2 \mapsto \mathsf{SO}_3$

3.4. *Consider some vector $\vec{x} \in \mathbb{R}^3$ and some real matrix $O \in \mathbb{R}^{3\times3}$. Write $\vec{x}' = O\vec{x}$ show that if $\vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y}$ this implies that $O^{-1} = O^\top$ and that if $\vec{x}' \times \vec{y}' \cdot \vec{z}' = \vec{x} \times \vec{y} \cdot \vec{z}$ then* $\det O = 1$, *hence $O \in SO_3$. Hint:* $\det A = \sum_{i_1,i_2,\ldots,i_n} \varepsilon_{i_1,i_2,\ldots,i_n} A_{1,i_1} A_{2,i_2} \ldots A_{n,i_n}$.

Solution:

$$
\begin{aligned}
\vec{x}' \cdot \vec{y}' &= \sum_i x_i' y_i' && \text{(17a)} \\
&= \sum_{i,j,k} O_{ij} O_{ik} x_j y_k && \text{(17b)} \\
&\overset{!}{=} \sum_i x_i y_i && \text{(17c)} \\
\iff & \sum_i O_{ij} O_{ik} = \delta_{j,k} && \text{(17d)} \\
\iff & O^T = O^{-1} && \text{(17e)}
\end{aligned}
$$

and

$$\vec{x}' \times \vec{y}' \cdot \vec{z}' = \sum_{i,j,k} \varepsilon_{ijk} x'_j y'_k z'_i \tag{18a}$$

$$= \sum_{i,j,k,i',j',k'} \varepsilon_{ijk} O_{ii'} O_{jj'} O_{kk'} x_{j'} y_{k'} z_{i'} \tag{18b}$$

$$\stackrel{!}{=} \sum_{i',j',k'} \varepsilon_{i'j'k'} x_{j'} y_{k'} z_{i'}, \tag{18c}$$

hence, $\sum_{i,j,k} \varepsilon_{ijk} O_{ii'} O_{jj'} O_{kk'} \stackrel{!}{=} \varepsilon_{i'j'k'}$. Choosing $i' = 1, j' = 2, k' = 3$ results in $\sum_{i,j,k} \varepsilon_{ijk} O_{i1} O_{j2} O_{k3} = \det O^\top = \det O^{-1} \stackrel{!}{=} \varepsilon_{123} = 1$, or thus $\det O = 1$.

3.5. *We construct ψ:*

(a) *Define $x := \sum_i x^i \sigma_i$, $\vec{x} \in \mathbb{R}^3$. This x is an element of $\mathfrak{su}(2)$!*
Show that $\vec{x} \cdot \vec{y} = \frac{1}{2} \operatorname{Tr}(xy)$, that $\vec{x} \times \vec{y}$ corresponds to $\frac{1}{2i}[x, y]$ and $\vec{x} \times \vec{y} \cdot \vec{z} = \frac{1}{4i} \operatorname{Tr}([x, y]z)$.

(b) *For $U \in \mathsf{SU}_2$, show that $\operatorname{Tr}\big(UxU^{-1}\big) = 0$ and $(UxU^{-1})^\dagger = UxU^{-1}$. Hence, $UxU^{-1} \in \mathfrak{su}(2)$.*

(c) *Now we show that the map induced by U, $O := \psi(U) : x \mapsto UxU^{-1}$, is in fact a special orthogonal transformation, $O \in \mathsf{SO}_3$. Concretely: $UxU^{-1} = \sum_i \tilde{x}^i \sigma_i$, where $\tilde{\vec{x}} = O\vec{x}$.*
Show that
   i. *$\vec{x} \cdot \vec{y}$ remains invariant under O,*
   ii. *same for $\vec{x} \times \vec{y} \cdot \vec{z}$.*

(d) *Now show that this map is a homomorphism: $\psi(U_1)\psi(U_2) = \psi(U_1 U_2)$.*

(e) *Now we need to show that ψ is surjective. We leave this as an extra exercise as this is quiet tedious to show. One approach to prove this, is to find explicit SU_2 matrices that get mapped to the generators of SO_3.*

(f) *Observe that $\psi(U) = \psi(-U)$ and show that the kernel of ψ is $\mathbb{Z}_2 \cong \{\pm \mathbb{1}_2\}$.*

(g) *Conclude that $\mathsf{SU}_2/\mathbb{Z}_2 \cong \mathsf{SO}_3$. Hint: use that $\mathsf{G}_1/\ker\varphi \cong im\varphi$, where $\varphi : \mathsf{G}_1 \mapsto \mathsf{G}_2$ is a homomorphism (to prove this is an extra exercise).*

Solution:

(a) We have that $\sigma_i \sigma_j = \delta_{ij} \mathbb{1}_2 + i \sum_k \varepsilon_{ijk} \sigma_k$ from which it follows that $\operatorname{Tr}(\sigma_i \sigma_j) = 2\delta_{i,j}$. Hence, $\frac{1}{2}\operatorname{Tr}(xy) = \frac{1}{2}\sum_{i,j} x^i y^j \operatorname{Tr}(\sigma_i \sigma_j) = \sum_i x^i y^i = \vec{x} \cdot \vec{y}$.
Since $[\sigma_i, \sigma_j] = 2i \sum_k \varepsilon_{ijk} \sigma_k$, we calculate that $\frac{1}{2i}[x, y] = \frac{1}{2i}\sum_{ij} x^i y^j [\sigma_i, \sigma_j] = \sum_{ijk} x^i y^j \varepsilon_{ijk} \sigma_k = \sum_{kij} \varepsilon_{ijk} x^i y^j \sigma_k = \sum_k (\vec{x} \times \vec{y})^k \sigma_k$.
$\vec{x} \times \vec{y} \cdot \vec{z} = \frac{1}{4i}\operatorname{Tr}([x, y]z)$ follows from the two previous identities.

(b) $\operatorname{Tr}\big(UxU^{-1}\big) = \operatorname{Tr}(x) = 0$ by virtue of the cyclicity of the trace and the fact that the Pauli matrices are traceless.
$(UxU^{-1})^\dagger = (U^{-1})^\dagger x^\dagger U^\dagger = UxU^{-1}$ by virtue of the fact that U is unitary and the Pauli matrices are Hermitian.

(c) i. To show that the inner product is invariant, we use the results from (3.5a) and (3.5b):

$$\vec{x}' \cdot \vec{y}' = \frac{1}{2}\operatorname{Tr}\big(x'y'\big) \tag{19a}$$
$$= \frac{1}{2}\operatorname{Tr}\big(UxU^{-1}UyU^{-1}\big) \tag{19b}$$
$$= \frac{1}{2}\operatorname{Tr}(xy) \tag{19c}$$
$$= \frac{1}{2}\vec{x} \cdot \vec{y} \tag{19d}$$

ii. Analogously:

$$\vec{x}' \times \vec{y}' \cdot \vec{z}' = \frac{1}{4i}\operatorname{Tr}\big([x', y']z'\big) \tag{20a}$$
$$= \frac{1}{4i}\operatorname{Tr}\big(U[x, y]U^{-1}UzU^{-1}\big) \tag{20b}$$
$$= \frac{1}{4i}\operatorname{Tr}([x, y]z) \tag{20c}$$
$$= \vec{x} \times \vec{y} \cdot \vec{z} \tag{20d}$$

(d) We act with $\psi(U_1)\psi(U_2)$ on some vector x and compute $\psi(U_1)\psi(U_2)x = \psi(U_1)\left(U_2 x U_2^{-1}\right) = U_1 U_2 x U_2^{-1} U_1^{-1} = U_1 U_2 x \left(U_1 U_2\right)^{-1} = \psi(U_1 U_2)x$.

(e) It is clear that $\psi(-U)x = (-U)x(-U)^{-1} = UxU^{-1} = \psi(U)x$. It is also clear that $\psi(\mathbb{1})x = x$ such that $\mathbb{1}$ and thus also $-\mathbb{1}$ constitute the kernel of ψ. It is straightforward to see that $\pm\mathbb{1}$ can be identified with $\mathbb{Z}_2$.

(f) Since $\psi : \mathsf{SU}_2 \mapsto \mathsf{SO}_3$ is surjective we have that $\mathrm{im}\psi = \mathsf{SO}_3$. Using the result from (3.5e), we can apply the theorem to obtain the desired result that $\mathsf{SU}_2/\mathbb{Z}_2 \cong \mathsf{SO}_3$.

Groups & Representations
Exercise session 8

May 20, 2022
16:00-17:30

1 Tensor product of representations of Lie algebras

1.1. *Suppose* (ρ_1, V_1) *and* (ρ_2, V_2) *are representations of a Lie algebra* $\mathfrak{g}$. *Show that*

$$(\rho_1 \otimes \rho_2)(X) = \rho_1(X) \otimes \mathbb{1} + \mathbb{1} \otimes \rho_2(X) \tag{1}$$

defines a representation on $V_1 \otimes V_2$, *the* tensor product representation *of* (ρ_1, V_1) *and* (ρ_2, V_2), *denoted by* $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$.

Solution: We will denote $\rho := \rho_1 \otimes \rho_2$. We thus need to show that $\rho([X,Y]) = [\rho(X), \rho(Y)]$ given that ρ_1 and ρ_2 satisfy $\rho_i([X,Y]) = [\rho_i(X), \rho_i(Y)], i = 1, 2$ (they form representations).

We write out $[\rho(X), \rho(Y)]$:

$$\begin{aligned}
[\rho(X), \rho(Y)] &=: [\rho_1(X) \otimes \mathbb{1} + \mathbb{1} \otimes \rho_2(X), \rho_1(Y) \otimes \mathbb{1} + \mathbb{1} \otimes \rho_2(Y)] && \text{(2a)} \\
&= (\rho_1(X) \otimes \mathbb{1} + \mathbb{1} \otimes \rho_2(X)) (\rho_1(Y) \otimes \mathbb{1} + \mathbb{1} \otimes \rho_2(Y)) \\
&- (X \leftrightarrow Y) && \text{(2b)} \\
&= \rho_1(X)\rho_1(Y) \otimes \mathbb{1} + \rho_1(X) \otimes \rho_2(Y) + \rho_1(Y) \otimes \rho_2(X) + \mathbb{1} \otimes \rho_2(X)\rho_2(Y) \\
&- (X \leftrightarrow Y) && \text{(2c)} \\
&= \rho_1(X)\rho_1(Y) \otimes \mathbb{1} + \mathbb{1} \otimes \rho_2(X)\rho_2(Y) - (X \leftrightarrow Y) && \text{(2d)} \\
&= [\rho_1(X), \rho_1(Y)] \otimes \mathbb{1} + \mathbb{1} \otimes [\rho_2(X), \rho_2(Y)] && \text{(2e)} \\
&= \rho_1 ([X,Y]) \otimes \mathbb{1} + \mathbb{1} \otimes \rho_2 ([X,Y]) && \text{(2f)} \\
&=: (\rho) ([X,Y]) . && \text{(2g)}
\end{aligned}$$

We used the usual properties of the tensor product and the fact that the ρ_i's form a representation.

2 Tensor product of representations of $\mathfrak{su}(2)$

2.1. *Show that* $\dim j_1 \otimes j_2 = \dim \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} j$.

Hint: assume w.l.o.g $j_1 \geq j_2$ *and observe that* $\dim \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} j$ *is an arithmetic sequence.*

2.2. Compute the tensor product of two spin 1/2 representations and of three spin 1/2 representations.

2.3. Show that every highest weight representation of $\mathfrak{su}(2)$ *can be obtained by taking tensor products of the* fundamental representation **2**.

2.4. In this exercise we will numerically investigate the tensor product representation $\mathbf{2} \otimes \mathbf{2}$.

(a) Consider the adjoint (spin 1) representation of $\mathfrak{su}(2)$ *from the previous exercise session:*

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tag{3a}$$

$$J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tag{3b}$$

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3c}$$

Go to the so called spherical basis $R_i = U J_i U^\dagger$ *with the basis transformation*

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}. \tag{4}$$

(b) Define an arbitrary element of $\mathfrak{su}(2)$ *in the spin 1/2 representation:* $F = x\frac{\sigma_X}{2} + y\frac{\sigma_Y}{2} + z\frac{\sigma_Z}{2}$, *with arbitrary* $x, y, z \in \mathbb{R}$.

(c) Construct the tensor product representation $\mathbf{2} \otimes \mathbf{2}$, $F \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes F$.

(d) Implement the basis transformation

$$V = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

and show that in this basis the tensor product representation falls apart in a trivial block and a three-dimensional representation, ie. $V(F \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes F)V^\dagger = 0 \oplus A$. *Where the three-dimensional adjoint representation reads* $A = xR_x + yR_y + zR_z$.

Solution:

2.1. The left-hand side reads $\dim j_1 \otimes j_2 = \dim j_1 \dim j_2 = (2j_1 + 1)(2j_2 + 1)$.

The right-hand side gives

$$\dim \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} j = \sum_{j=|j_1-j_2|}^{j_1+j_2} \dim j \tag{6a}$$

$$= \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1). \tag{6b}$$

This is a arithmic sequence as this can be written as (assume that $j_1 > j_2$):

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = (2(j_1 - j_2) + 1) + (2(j_1 - j_2) + 2 + 1) + ... + (2(j_1 + j_2) + 1). \tag{7}$$

Which is equal to $\frac{1}{2}(2(j_1 - j_2) + 1 + 2(j_1 + j_2) + 1)(2j_2 + 1) = (2j_1 + 1)(2j_2 + 1)$ as j takes $2j_2 + 1$ possible values.

2.2. We find that $\mathbf{2}\otimes\mathbf{2} = \mathbf{1}\oplus\mathbf{3}$. Using this we compute that $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = \mathbf{2}\otimes(\mathbf{1}\oplus\mathbf{3}) = \mathbf{2}\oplus\mathbf{2}\oplus\mathbf{4}$.

The tensor product of two spinor representations ($\mathbf{2}$, $j = 1/2$) thus decomposes in a spin singlet ($\mathbf{1}$) and spin triplet ($\mathbf{3}$). Three spin 1/2 representations decompose in 2 $\mathbf{2}$'s and a four-dimensional irrep ($\mathbf{4}$, $j = 3/2$).

2.3. Consider the tensor product of two $\mathbf{2}$'s (fundamental representation): $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$, from this we find the trivial ($\mathbf{1}$) and vector ($\mathbf{3}$) representation. Extracting from this decomposition the three-dimensional block $\mathbf{3}$ and tensoring with another fundamental representation $\mathbf{2}$ gives us the $\mathbf{4}$ as $\mathbf{2} \otimes \mathbf{3} = \mathbf{2} \oplus \mathbf{4}$. Tensoring the $\mathbf{4}$ with another $\mathbf{2}$ gives us the $\mathbf{5}$ and so on. Continuing like this we find all the irreps of $\mathfrak{su}(2)$: $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, ...$ by only tensoring with the fundamental representation. Hence, the name *fundamental*: all irreps can be constructed starting from this two-dimensional representation given by the Pauli matrices (see previous exercise session).

2.4. An implementation can be found here.