Group and representation theory: A Shitty summary

Chapter 2: Groups: Basic definitions and properties

2.1) Basics: finite groups

def: A group is a set of elements which can transform into eachother associatively $[(a\cdot b)\cdot c = a\cdot(b\cdot c) \dots]$ via a group action (not the GA) $\phi: G\times G \to G$. A group must contain: an identity element, and every element must have an inverse: $\forall g \in G, \exists g^{-1} \in G: g\cdot g^{-1} = g^{-1}g = 1$

def: we can define 2 types of group actions

Let G be a group, X be some finite set

Left group action:

$\alpha: G\times X \to X$ if α satisfies: $\forall x \in X: \alpha(e, x) = x$ (identity)

~~$\forall g,h,x\in X$~~: $\alpha(g, \alpha(h,x)) = \alpha(gh, x)$ $\forall g,h \in G, x \in X$

we say the group acts on X (from the left).

A set X combined with some action of G is called a (left) G-set.

Right group action:

$\alpha: X\times G \to X$ if α satisfies:

$\forall x \in X: \alpha(x, e) = x$

$\forall g,h \in G, x\in X: \alpha(\alpha(x,g),h) = \alpha(x,$

if the action is commutative then $LA = RA$.

We can say the action of a group G on some set X is:

transitive: $\forall x,y \in X, \exists g \in G: g\cdot x = y$

faithful: $\forall g,h \in G, \exists x \in X: g\cdot x \neq h\cdot x$

free: $\forall g,h \in G, x\in X: g\cdot x = h x \Rightarrow g = h$ (free action is always faithful)

regular: transitive + free

⋮

the number of elements in a group is called it's cardinality. (denoted #G or |G|)

we can display the way 2 group elements combine through their action with a cayley table.

let $\{a, b, c\}$ be some group, it's cayley table is then

$\cdot$	a	b	c
a	a^2	ab	ac
b	ba	b^2	bc
c	ca	cb	c^2

we can derive some interesting properties from this table:

abelian group $\Leftrightarrow$ cayley table is symmetric along its diagonal

a very important family of groups is the permutation group S_n of n elements. it has cardinality $n!$ and is also called the symmetric group.

we can also choose to define our groups through a generating set of group elements, that through multiplication, result in all the elements of that group

example: the dihedral group of order n is generated by (r, s) where r is a rotation over π/n and s is any reflection about a line of symmetry.

the upper limit on the size of this set is $\log_2(\#G)$, where this is only satisfied by a generating set created by direct products of $\mathbb{Z}_2$'s

2.1.2) Automorphisms

def:

an automorphism of a group is a permutation (read: relabeling) of its elements $a: G \to G$ s.t the cayley table remains invariant under that relabeling example:

the regular cayley table of the Klein four-group (every element is its own inverse):

$$\begin{matrix} e & a & b & c \\ a & e & c & b \\ b & c & e & a \\ c & b & a & e \end{matrix} \xrightarrow{f} \begin{matrix} e & c & a & b \\ c & e & b & a \\ a & b & e & c \\ b & a & c & e \end{matrix}$$

and the second is under some automorfism f given by:

$$f(e) = e$$
$$f(a) = c$$
$$f(b) = a$$
$$f(c) = b$$

another way to define the automorfism is by saying:

$f \in Aut(G)\ \forall\ g_1, g_2 \in G : f(g_1) f(g_2) = f(g_1 g_2)$

the automorphisms of a group G themselves also form a group

2.1.3) Classes and Subgroups

def:

A subgroup S of some group G is a subset of elements of G that themselves form a group.

for some subgroup H of some group G we define the left coset $aH = \{ah \mid h \in H\}$ for some $a \in G$. the amount of left cosets is denoted by $[G:H]$ and is called the index. (analogous for right cosets).

Theorem:

Lagrange's theorem states that for some subgroup H of G:

$$[G:H] = \frac{|G|}{|H|}$$

(thus that the cardinality of H is a divisor of the cardinality of G)

Proof:

the left cosets are equivalence classes of some relation on G. call x and y in G equivalent if $\exists h \in H$ s.t. $x = hy$. the left cosets thus define a partition of G! each left coset aH must have the same cardinality as H because $x \to ax$ defines a bijection $H \to aH$. thus

$$|G| = [G:H] \cdot |H|$$

def:

the centre of some group G is the subgroup of elements which commute with all other elements of G. (Also called the centraliser, denoted $Z(G)$)

(trivially found by looking for zero rows in $C - C^T$ where C is the cayley table.)

another way of constructing interesting subgroups is by creating classes through conjugation: for some g we create the set $\{hgh^{-1} \mid h \in G\}$. we can divide the group in such classes and every element is contained in exactly 1 class.

if we now create a subgroup of complete classes this defines a group closed under conjugation, which we call a normal/invariant subgroup (note the centre is a normal subgroup)

a subgroup N is normal if thus it is invariant under conjugation with all group elements of its 'parent' group. we also have $\forall g \in G: gN = Ng$.

this gives rise to some types of groups:

- simple:

  a group is called simple if it's only normal subgroup is the set consisting of solely the identity $\{e\}$

- factor:

  the factor (or quotient) group G/N is the group of cosets aN

- maximal normal subgroup:

  a normal subgroup that contains no non trivial subgroups (it's factor group is thus simple)

- commutator group:

  this is the group generated from all group elements $xyx^{-1}y^{-1}$. this allows us to define the Abelianization $G/[G,G]$.

2.7.4) extending groups

Semi-direct product

for two groups N and H the semi-direct product of N and H w.r.t β

$$G = N \rtimes_\beta H$$

is the group of elements $N \times H$ i.e all elements of G are as (n,h) with group multi.
defined as

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1 \beta_{h_1}(n_2), h_1 h_2) \qquad (\beta_{h_1} = \beta(h_1))$$

with β a homomorfism $\beta: H \to Aut(N)$

N then is a normal subgroup of G (H not necessarily, but it is always a subgroup)

(note that this only defines G up to isomorfism)

Sequences and extensions (go ahead and skip this)

def:

an exact sequence is a sequence of homomorfisms where the image of one morfism is the kernel of the next. i.e

$$im(\hat{\phi}_i) = ker(\hat{\phi}_{i+1}) \quad \forall i$$

$(\phi_i : G_i \to G_{i+1})$

def:

a short exact sequence is an exact sequence of the form

$$1 \to A \xrightarrow{f} B \xrightarrow{g} B/A \to 1$$

a special type of short sequence is called a split short sequence.
this is when $\exists\, h: B/A \to B : g \circ h = id_{B/A}$

A good question to ask now is, for some given N and H find all G compatible with

$$1 \to N \to G \to H \to 1$$

as it turns out this is a very difficult problem. but if we demand that this is a split short sequence, then all constructions are found through the semi direct product:

$$N \rtimes H$$

we call these the splitting extensions

another type of interesting extension is the central extension:

def:

central extensions G of some group H by an Abelian group A are defined by:

$$1 \to A \to G \to H \to 1$$

where we require A to be a subgroup of the centralizer $Z(G)$ and where $H = G/A$

all central extensions can be found by defining elements (a,h) and a group multiplication rule:

$$(a_1, h_1) \cdot (a_2, h_2) = (a_1 a_2 + \omega(h_1, h_2), h_1 h_2).$$

Rest of chapter 2 talks about cohomology. This is not expected to be known.

Chapter 3: Representation theory.

up till now i have been rewriting / clarifying the course notes doc. seeing as how 90% of it isn't expected to be known i shall now base myself off of the theory included in the exercise session documents.

3.1) definition and left/right regular repr.

def:

a representation of some finite group G is a group homomorfism $\rho : G \to GL(V)$, where V is a finite vector space. i.e. ρ maps every group element g to some matrix X_g s.t

$$X_{g_1} X_{g_2} = X_{(g_1 \cdot g_2)} \quad \forall g_1, g_2 \in G$$

we define the dimension of the representation as $\dim(V)$

we call a representation faithful if $\mathcal{S}$ is injective (every element has atleast 1 image value)

And now, Back to the lecture notes :

note that $\mathcal{S}(e) = \mathbb{1}$. this implies that all these matrices X_g must be invertible !
even stronger still: for finite groups we will always be able to restrict ourselves to
unitary matrices ($A^{-1} = A^{\dagger}$)

finally, it holds that for any 2 equivalent representations (that is to say, 2 repr. related through conjugation : $A_g = X D_g X^{-1}$) they are related by a unitary conjugation

now to construct these representations.
one clear way is by using the regular representation, which is obtained by constructing $n \times n$ permutation matrices which exactly encode the information of the cayley table (thus how the elements transform into eachother) of the group with cardinality n.

Left regular:
the left regular representation is given by

$$L_g = \sum_h |g \cdot h\rangle\langle h|$$

for which $L_x \cdot L_y = L_{xy}$.

$|g\rangle$: groups element
$|g \cdot h\rangle\langle h|$ is $h \to g \cdot h$ en doet niets met andere elem

Right regular :
this is given by

$$R_g = \sum_h |h \cdot g\rangle\langle h|$$

for which $R_x R_y = R_{y \cdot x}$

3.0.7) irreducible representations: Basics

def: invariant subspace:
A subspace Y is called (left/right) invariant if $\forall A_g$ in the algebra

$$A_g Y \subset Y \quad \text{(left-invariant)}$$

$$Y A_g \subset Y \quad \text{(right-invariant)}$$

for matrix-algebras with invariant subspaces there always exists a basis in which we can write these matrices as block diagonal

$$A(z) = \begin{bmatrix} B(z) & D(z) \\ 0 & C(z) \end{bmatrix} \text{ with } \begin{bmatrix} X \\ 0 \end{bmatrix} \quad \text{(left-inv)}$$

$$A(z) = \begin{bmatrix} B(z) & 0 \\ D(z) & C(z) \end{bmatrix} \quad \text{(right-inv)}$$

there now exists a theorem (Maschke's theorem) that says:

"There always exists a similarity transform ($P^{-1}AP$) which makes all the matrices of the representation block diagonal"

proof:

starting from: $A(z) = \begin{pmatrix} B(z) & 0 \\ D(z) & C(z) \end{pmatrix}$

we want:

$$\begin{pmatrix} I & 0 \\ -F & I \end{pmatrix} \begin{pmatrix} B(z) & 0 \\ D(z) & C(z) \end{pmatrix} \begin{pmatrix} I & 0 \\ F & 0 \end{pmatrix}$$

to be diagonal.

this is equal to: $\begin{pmatrix} B(z) & 0 \\ -FB(z) + D(z) + C(z)F & C(z) \end{pmatrix}$

or simply:

$$-FB(z) + D(z) + C(z)F = 0$$

let $F = \frac{1}{|G|} \sum_a D(a) B(a^{-1})$

then:

$$\frac{-1}{|G|} \sum D(a) B(a^{-1}) B(z) + D(z) + \frac{1}{|G|} \overset{C(z)}{\sum} D(a) B(a^{-1})$$

but

$$D(za) = D(z) B(a) + C(z) D(a) \leftrightarrow C(z) D(a) = D(za) - D(z) B(a)$$

substituting this we find

$$0 \overset{?}{=} \frac{-1}{|G|} \sum D(a) B(a^{-1}) B(z) + D(z) + \frac{1}{|G|} \sum \left(D(za) - D(z) B(a)\right) B(a^{-1})$$

$$= \frac{-1}{|G|} \sum_a D(a) B(a^{-1}) D(z) + D(z) + \frac{1}{|G|} \sum D(za) B(a^{-1}) - \underbrace{\frac{1}{|G|} \sum_a D(z)}_{D(z)} \underbrace{B(a) B(a^{-1})}_{1}$$

$$= \frac{-1}{|G|} \sum_a D(a) D(a^{-1}) B(z) + \frac{1}{|G|} \sum D(za) B(a^{-1})$$

where the two sums are equal because of the rearrangement theorem □

and thus we have proven it

thus any representation $U(g)$ can be written in which

$$U(g) = \begin{pmatrix} U_1(g) & 0 & \cdots & \vdots \\ 0 & U_2(g) & & \vdots \\ \vdots & 0 & \ddots & \\ & \vdots & & \end{pmatrix}$$

where the $U_i(g)$'s dont have invariant subspaces !

Def: irreducible representation (irreps):

an irrep is a matrix representation of a group with no invariant subspaces.

a crucial tool will be schur's lemma's :

irrep span matrix-alg. → 1: if $U(g)$ is an irrep and $\forall g : X U(g) = U(g) X \Rightarrow X = c . I$

2: given 2 irreps $U(g), V(g)$ and a matrix T s.t. $\forall g : T U(g) = V(g) T$ then U, V equivalent or $T = 0$

for finite groups, any irrep is always equivalent to an irrep of unitary matrices

Proof:
seek lecture notes (is not expected to be known)

a second point is that representations can be written as a direct sum of irreps this as a direct consequence of maschke's theorem. block diagonalisable → ⊕ irreps

The great orthogonality theorem

Consider all N inequivalent unitary irreps of G: $D^{\alpha}_{ij}(x)$ (d_α = matrix dim.)
(α labels the irreps). Then:

$$\frac{1}{|G|}\sum_x D^{\alpha}_{ij}(x)\,\bar{D}^{\beta}_{kl}(x) = \frac{1}{d_\alpha}\delta_{\alpha\beta}\delta_{ik}\delta_{jl}$$

(proof not expected to be known)

This implies:
$$\sum_{\alpha}^{\hat{N}} d_\alpha^2 \leq |G|$$

Proof:
Let $i=j$, $k=l$ in G.O.T

$$\frac{1}{|G|}\sum_x D^{\alpha}_{ii}(x)\bar{D}^{\beta}_{kk}(x) = \frac{1}{d_\alpha}\delta_{\alpha\beta}\delta_{ik}$$

rewritten:

$$\frac{1}{|G|}\sum_x\sum_i\sum_k D^{\alpha}_{ii}\,\bar{D}^{\beta}_{kk} = \frac{1}{d_\alpha}\delta_{\alpha\beta}\sum_i\sum_k\delta_{ik} = \delta_{\alpha\beta}$$

sums over i and k are the traces of D^α and $\bar{D}^\beta$, which we define as the characters of these representations

$$\frac{1}{|G|}\sum\sum\sum D^{\alpha}_{ii}\,\bar{D}^{\beta}_{kk} = \frac{1}{|G|}\sum_x \mathrm{tr}(D^\alpha(x))\,\overline{\mathrm{tr}(D^\beta(x))} = \underbrace{\frac{1}{|G|}\sum_x \chi^\alpha(x)\bar{\chi}^\beta(x) = \delta_{\alpha\beta}}_{\text{G.O.T for characters}}$$

Characters of representations

where irreps are rather arbitrary (defined up to equivalence relation)
their traces are basis invariant, and therefore universal we thus define

$$\boxed{\chi^\alpha(x) = \mathrm{Tr}(D^\alpha(x))}$$

the group character is constant in a given class (i.e conjugation with other group elements) Let us denote the classes as C_k and define N_k as the number of elements in that class

from before we know, the G.O.T for characters:

$$\frac{1}{|G|}\sum_x \chi^{(\alpha)}(x)\,\chi^{(\beta)}(x) = \delta_{\alpha\beta}$$

they thus form an orthonormal basis in the space of group elements we can construct a matrix with orthonormal columns $O_{\lambda\alpha}$:

$$\sqrt{\frac{N_\lambda}{|G|}}\, \chi^\alpha(A_\lambda)$$

which shows:

- $\# C_\lambda \geq |G|$ (otherwise $O_{\lambda\alpha}$ not full rank)
- if 2 irreps have the same characters → they are equivalent

we can thus readily calculate how many times a reducible rep. contains an irrep : calculate the inner prod of the corresponding characters

in particular for the left-regular repr. say an irrep λ:

$$c_\lambda = \frac{1}{|G|}\sum_x \chi^\lambda(x)\, \bar{\chi}^L(x) = d_\lambda \qquad \left(L(g) = \sum_h |g\cdot h\rangle\langle h|\right)$$

(d_α = dim of irrep α)

notice $\chi^\lambda(1) = d_\lambda$. this now implies that the left-regular representation contains each d_α dim-irrep $n_\alpha = d_\alpha$ times!

this now implies

$$\sum_\alpha d_\alpha^2 = |G|$$

and that $\sum c_\lambda = |G|$

see text for example.

Real, complex and quaternionic representations, the schur indicator and Kramer's theorem

We can classify irreps into 3 categories. By defining an anti-unitary operator J for which : $J(\sum \alpha_i x_i) = \sum \bar{\alpha}_i J(x_i)$. we define the complex conjugate repr. $\bar{U}$ of U as:

$$\bar{U}(g) = J\, U(g)\, J \qquad (J^2 \text{ is } I \to \text{is also a repr})$$

a repr is self conjugate if $\bar{U}$ is equivalent to U. i.e : $\exists X: \bar{U} = XUX^{-1}$

if there exists no such X → repr is complex taking the conjugate of

$$\bar{U} = X U X^{-1}$$

gives

$$U(g) = \bar{X} X \, U(g) \, (\bar{X} X)^{-1} \quad (1)$$

schurs lemma says $\bar{X} X = c \cdot I$ and $X = c X^{T}$

taking the conjugate of (1) then

$$U(g) = c^2 \, \bar{U}(g)$$

remember U to be self conjugate s.t $c^2 = 1! \to c = \pm 1$ which distinguishes the 2 other classes:

real, $c = +1$: $X = X^T$, thus with some decomposition we write $X = Q Q^T$. we want some Y s.t: $\overline{Y U(g) Y^{-1}} = Y U(g) Y^{-1}$ or $Y X U X^{-1} Y^{-1} = Y U Y^{-1}$. $Y = Q^T$ is just that.

quat: $c = -1$:

wont be asked, wont write it down

a simple way to verify what type of repr you have is throug its characters

$$\frac{1}{|G|} \sum_{x} \chi(x^2) = \begin{cases} 1 & \text{real} \\ 0 & \text{complex} \\ -1 & \text{quat} \end{cases}$$

proof:

$$\frac{1}{|G|} \sum_{x} \chi(x^2) = \frac{1}{|G|} \sum_{ijx} U_{ij}(x) \, U_{ji}(x) = \frac{1}{|G|} \sum_{ijx} U_{ij}(x) \, \bar{\bar{U}}_{ji}(x)$$

if U is real the sum reduces to $\frac{1}{d_i} \sum \delta_{ij} \delta_{ji} = 1$

if U is complex, the repr are inequivalent $\to \sum = 0$

if U is quat, $U = X^{+} \bar{U} X$, $\bar{X} X = -I$, sum becomes

$$\frac{1}{|G|} \sum tr(X^{+} \bar{U}(x) X \bar{\bar{U}}(x)) = \frac{1}{d} \sum \bar{X}_{ij} X_{ji} = \frac{1}{d} tr(\bar{X} X) = -1$$

Projective representations

the central idea here is that if $U(g) \otimes U(g)$ forms a representation, $U(g)$ does not always form a repr. if they satisfy

$$U(g)U(h) = e^{i\omega(g,h)} U(gh) \qquad \omega(g,h) \text{ some real phase}$$

then $U \otimes U$ is a repr but U isnt, we call U a projective repr.
a general projective repr is defined as one where its product is not compatible with the group action but results in an extra prefactor

$$D(g)D(h) = f(g,h)\, D(g,h)$$

we impose the condition on ω that:

$$\omega(x,y) + \omega(x,y,z) =_{2\pi} \omega(x,y,z) + \omega(y,z) \quad (2) \qquad (=_{2\pi} \text{ means equivalent in } \bmod 2\pi)$$

this implies there are only a finite amount of projective repr!

we will derive (2):

$$U(x)U(y)U(z) = U(x)\left(e^{i\omega(y,z)} U(yz)\right) = e^{i\omega(x,yz) + i\omega(y,z)} U(x,y,z)$$

en representatie is altijd associatief zodus:

$$U(x)\left[U(y)U(z)\right] = \left[U(x)U(y)\right]U(z) = e^{i\omega(x,y) + i\omega(xy,z)} U(xyz)$$

dus moet

$$\omega(x,yz) + \omega(y,z) =_{2\pi} \omega(x,y) + \omega(xy,z) \qquad (e^{ix} = e^{i(x+2\pi)})$$

Constructing Projective repr

for some given cocycle $\omega(g,h)$ we find its corresponding projective repr by starting from the regular representation

$$R_\omega = \sum_h e^{i\omega(g,h)} |g \cdot h\rangle\langle h|$$

indeed: $\quad$ $\uparrow$ $x = h y$, ander $=0$ (orthonormal)

$$R_\omega(g) R_\omega(h) = \sum_{x,y} e^{i(\omega(g,x) + \omega(h,y))} |g\cdot x\rangle\langle x|h\cdot y\rangle\langle y|$$

$$= \sum_{y} e^{i(\omega(g,h\cdot y) + \omega(h,g))} |g\cdot h\cdot y\rangle\langle y|$$

$$= \sum_{y} e^{i(\omega(g\cdot h,y) + \omega(g,h))} |g\cdot h\cdot y\rangle\langle y|$$

$$= e^{i\omega(g,h)} R_\omega(g\cdot h)$$

note, projective representations with non-trivial cocycles can't have 1 dim subspaces. Otherwise, let $|v\rangle$ be this inv. subspace:

$$R_\omega(g)\overset{|v\rangle}{\uparrow} = e^{i\alpha(g)} |v\rangle$$

$$R_\omega(g) R_\omega(h) = e^{i(\alpha(g) + \alpha(h))} |v\rangle$$
$$= e^{i\omega(g,h)} R(g\cdot h) |v\rangle$$
$$= e^{i\omega(g,h)} e^{i\alpha(gh)} \cancel{R_\omega(gh)}|v\rangle$$

$\downarrow$ dit volgt niet uit dit, is gewoon ook gelijk aan $R_\omega(g) R_\omega(h)$ uit def proj repr!!

now $\omega(g,h) = \alpha(g) + \alpha(h) - \alpha(gh)$ which is a trivial cocycle and thus a contradic.

3.0.2: The group algebra
3.0.3: young tableaux
3.0.4: induced reps

no theory to be known here, if needed for some exercise i will write it here

Chapter 4: Lie groups and their representations

4.1: Associativity

we would like to generalize now to continuous groups.
we do this by looking at group elements in a certain domain and impose that the group structure is encoded in an analytical function which satisfies associativity

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c)$$

4.2: Lie groups and structure factors

we require the associativity mentioned above, as well as unity elements and the existance of inverses.

to develop the theory of Lie groups we define a vector of variables x on which the elements of the group will act. The point of a lie group then is that its properties are completely controlled by behaviour under infinitesimal transformations

$$x' = f(x,a)$$
$$x' + dx' = f(x', \delta a)$$
$$= f(x, a + da)$$

clearly

$$da_i = \underbrace{\frac{\partial \phi_i(a,b)}{\partial b_j}\bigg|_{b=0}}_{\Theta_{ij}(a)} \delta a_j$$

we can invert this to get $\Psi(a)$

$$\delta a = \Psi(a)\, da$$

also

$$dx_i' = \sum_k \underbrace{\frac{\partial f_i(x',a)}{\partial a_k}\bigg|_{a=0}}_{\rightarrow\, U_{ik}(x')} \delta a_k$$

$$dx_i' = U(x')\,\Psi(a)\,da$$

$$\frac{dx_i'}{\delta a_k} = \sum_j U_{ij}(x')\,\Psi_{jk}(a)$$

looking now at some infinitesimal transformation

$$\begin{aligned} x_i + dx_i &= f_i(x, \delta a) \\ &= x_i + \sum_k \frac{\partial f_i}{\partial a_k}\delta a_k \\ &= x_i + \sum_k U_{ik}\,\delta a_k \\ &= \Big(1 + \sum_k \underbrace{\sum_i U_{ik}\frac{\partial}{\partial x_i}}_{\hat{X}_k}\delta a_k\Big)x_i \\ &= \Big(1 + \sum_k \hat{X}_k\,\delta a_k\Big)x_i \end{aligned}$$

where we defined the generator $\hat{X}_k$:

$$\boxed{\hat{X}_k = \sum_i U_{ik}(x)\frac{\partial}{\partial x_i}\quad,\quad U_{ik}(x) = \frac{\partial f_i(x,a)}{\partial a_k}\Big|_{a=0}}$$

we will study Lie groups through these generators and their commutator algebra

example 1:

$$x' = ax + b$$

$$\hat{X}_a = \frac{\partial(ax+b)}{\partial a}\cdot\frac{\partial}{\partial x} = x\frac{\partial}{\partial x} \qquad \hat{X}_b = \frac{\partial(ax+b)}{\partial b}\cdot\frac{\partial}{\partial x} = 1\frac{\partial}{\partial x}$$

$$[\hat{X}_a, \hat{X}_b] = x\cdot\frac{\partial}{\partial x}\left(1\frac{\partial}{\partial x}\right) - 1\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) = 0 - \frac{\partial}{\partial x} = -\hat{X}_b$$

example 2:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{cases} f_1 = ax_1 + bx_2 = x_1' \\ f_2 = cx_1 + dx_2 = x_2' \end{cases}$$

$$\hat{X}_a = \frac{\partial f_1}{\partial a}\frac{\partial}{\partial x_1} + \frac{\partial f_2}{\partial a}\frac{\partial}{\partial x_2} = x_1\frac{\partial}{\partial x_1} \qquad \hat{X}_c = \frac{\partial f_1}{\partial c}\cdot\frac{\partial}{\partial x_1} + \frac{\partial f_2}{\partial c}\cdot\frac{\partial}{\partial x_2} = x_1\frac{\partial}{\partial x_2}$$

$$\hat{X}_b = \frac{\partial f_1}{\partial b}\frac{\partial}{\partial x_1} + \frac{\partial f_2}{\partial b}\cdot\frac{\partial}{\partial x_2} = x_2\frac{\partial}{\partial x_1} \qquad \hat{X}_d = \frac{\partial f_1}{\partial d}\cdot\frac{\partial}{\partial x_1} + \frac{\partial f_2}{\partial d}\cdot\frac{\partial}{\partial x_2} = x_2\frac{\partial}{\partial x_2}$$

also see the final example on p 47.

4.2.1) Structure factors

to derive these, let us start from

$$\frac{\partial x_i}{\partial a_\ell} = \sum_j U_{ij}(x)\, \Psi_{j\ell}(a) \qquad (3)$$

where $x_i + dx_i = f_i(x, da)$.

all these functions are analytic, s.t:

$$\frac{\partial^2 x_i}{\partial a_\lambda \partial a_\mu} = \frac{\partial^2 x_i}{\partial a_\mu \partial a_\lambda}$$

note:

$$\frac{\partial U_{i\ell}(x)}{\partial a_\lambda} = \frac{\partial U_{i\ell}(x)}{\partial x_j}\frac{\partial x_j}{\partial a_\lambda} \overset{(3)}{=} \frac{\partial U_{i\ell}(x)}{\partial x_j} U_j^{\,\nu}(x)\, \Psi_{\nu\lambda} \qquad \text{(einst. summ)}$$

thus

$$U_{i\ell}(x)\left(\frac{\partial \Psi_{\ell\mu}}{\partial a_\lambda} - \frac{\partial \Psi_{\ell\lambda}}{\partial a_\mu}\right) = -\left(U_{j\nu}\frac{\partial U_{i\ell}}{\partial x_j} - U_{j\ell}\frac{\partial U_{i\nu}}{\partial x_j}\right)\Psi_{\ell\mu}\Psi_{\nu\lambda}$$

or by using the inverse

$$U_{j\sigma}(x)\frac{\partial U_{i\tau}(x)}{\partial x_j} - U_{j\tau}\frac{\partial U_{i\sigma}}{\partial x_j} = \underbrace{\left(\frac{\partial \Psi_{\ell\mu}}{\partial a_\lambda} - \frac{\partial \Psi_{\ell\lambda}}{\partial a_\mu}\right)\Theta_{\mu\tau}\Theta_{\lambda\sigma}}_{f^{\ell}_{\tau\sigma}(a)} U_{i\ell}(x)$$

partial derivatives yield:

x_i :
$$\frac{\partial}{\partial x_m}\left(U_{j\sigma}(x)\frac{\partial U_{i\tau}(x)}{\partial x_j} - U_{j\tau}\frac{\partial U_{i\sigma}}{\partial x_j}\right) = f^{\ell}_{\tau\sigma}\frac{\partial}{\partial x_m}U_{i\ell}(x)$$

a_g :
$$\frac{\partial f^{\ell}_{\tau\sigma}(a)}{\partial a_g}U_{i\ell} = 0$$

U is full rank s.t $f^{\ell}_{\tau\sigma}$ has no dependance on the group element and is a constant! they control the commutation relationships

$$\boxed{[\hat{X}_j(x), \hat{X}_\sigma(x)] = \sum_k f^{k}_{j\sigma}\hat{X}_k(x)}$$

note following properties:

- the structure factors must be anti-symmetric
- they must obey the jacobi identity

9.2.2: adjoint representation and exponential map

describing all lie groups has reduced to finding solutions $f_{\alpha\beta}^{\gamma}$ that satisfy the jacobi identity

the representation theory consists then of finding all irreps X_α satisfying the lie commutator

$$[X_\alpha, X_\beta] = \sum_\gamma f_{\alpha\beta}^{\gamma} \hat{X}_\gamma$$

for given $f_{\alpha\beta}^{\gamma}$.

irreps have generators, turns out we can construct the set of generators from the structure factors. we call this solution the adjoint repr.

given an anti-symm $f_{ab}^{c} = -f_{ba}^{c}$ solution to the jacobi eq., the adj repr is defined as:

$$\hat{X}_g := (X_g)_{ij} = -f_{gi}^{j}$$

it's matrix elements thus are the structure factors!

9.3: classifying lie algebras

we restrict ourselves to semi-simple, compact lie algebras and groups, where:

semi-simple: direct sum of simple

simple lie alg: trivial set of operators closed under commutation

compact group: allows us to take the generators of the algebra as anti-symm

4.3.7) Cartan subalgebra and Weyl basis

given the adjoint representation of some lie algebra, is it always possib. to bring the N generators into some canonical form through linear combinations.

the cartan subalgebra is a maximal commutating algebra and its dimensionality r will be important.

given a set of r commutating operators H_i, we can simultaneously diag. them to obtain r diagonal matrices D_i. their diagonal elements fully characterise the Lie algebra. they have r zero entries and $N-r$ non-zero (complex) entries. we can thus define $N-r$ vectors with r entries namely $\alpha(i)$ the im part of $D_i(\alpha,\alpha)$. these vectors are the root-vectors notated as $\vec{\alpha}$.

these $\vec{\alpha}$ allow us to write the algebra in the canonical cartan-weyl form

$$[H_i, H_j] = 0$$
$$[H_i, E_\alpha] = \alpha(i) E_\alpha$$
$$[E_\alpha, E_\beta] = e_{\alpha\beta} E_{\alpha+\beta}$$
$$[E_\alpha, E_{\alpha}] = \sum_i \alpha(i) H_i$$

$E_{\alpha/\beta}$ = raising / lowering operators

$e_{\alpha\beta} = 0$ if $\alpha+\beta$ is not a root, some integer otherwise

these root vectors obey following symmetries:

- if α_a is a root, so is $-\alpha_a$ $\Rightarrow$ E_α cant be hermitian
- $[E_\alpha, E_\beta]$ is either een eigenvector with root $\alpha+\beta$, or zero
- if we let $\beta=-\alpha$, $[H_i, [E_\alpha, E_{-\alpha}]] = 0 \rightarrow [E_\alpha, E_{-\alpha}]$ is in the cartan subalgebra

$$[E_\alpha, E_{-\alpha}] = \sum_i e_i H_i$$

if H_i orthonorm:

$$e_i = \alpha(i)$$

a crucial insight now, $\forall \vec{\alpha}$ there exists a set of 3 operators in the complete algebra, which form a sub algebra of SU(2)

this means the generators E_β fall into different sets which transform according to an irrep of SU(2) under the adjoint of the subalgebra $J^\alpha_{z,\pm}$:

$$J^\alpha_z = \frac{1}{||\alpha||^2} \sum \alpha(i) H_i$$
$$J^\alpha_\pm = \frac{1}{||\alpha||} E_{\pm\alpha}$$

- a vector root has no multiplicities
- if $\alpha(i)$ is a root, $\lambda\alpha(i)$ is only a root for $\lambda = -1$

Highest weight and numerology of roots

we shall now construct an irrep. we pick a largest eigenvalue (highest weight) w of the cartan sub algebra and then apply lowering operators to construct the irrep. we say $\vec{\alpha} > \vec{\beta}$ if the first non-zero component of $\alpha - \beta$ is > 0

all weights $w(i)$ are defined as eigenvalues of H_i's

$$H_i |w\rangle = w(i) |w\rangle$$

starting from some $w(i)$ we apply the raising and lowering operators to obtain the eigenvals $w(i) \pm k\alpha$ with corresp eigenvectors $|k\rangle = E^{\ell}_{\pm\alpha} |w\rangle$

these $|k\rangle$ now have to span an irrep of SU(2)

we find

$$\frac{\vec{w}\vec{\alpha}}{||\alpha||^2} = -\frac{n}{2} \qquad \text{with } n \text{ some integer}$$

note the severe restriction on angles of root vectors!

this also implies Weyl-reflection symmetry

$$\vec{w} \leftrightarrow w - 2\frac{\vec{w}\vec{\alpha}}{||\alpha||^2}\vec{\alpha}$$

we could have done this with the adjoint repr. the eigenvectors then are the generators themselves and $\omega(i) = \alpha(i)$. thus for any 2 roots

$$\frac{\vec{\beta}\vec{\alpha}}{\|\alpha\|^2} = -\frac{n'}{2} \qquad \frac{\vec{\alpha}\vec{\beta}}{\|\beta\|^2} = -\frac{n''}{2}$$

which greatly restricts all possible roots as:

$$\frac{(\beta\alpha)^2}{\|\alpha\|^2\|\beta\|^2} = \cos^2(\theta_{\alpha\beta}) \quad \text{and} \quad \frac{\|\alpha\|^2}{\|\beta\|^2} = \frac{n''}{n'}$$

which results

n	θ	α^2/β^2
0	90	any ch.
1	60/120	1
2	45/135	1/2, 2
3	30, 150	1/3, 3
4	0	1

all that remains is to find vectors which satisfy these conditions

See p49 for the example on $SO(3)$ (also see the root diagrams)

dynkin diagrams

theory questions end here :)

there is redundancy in root diagrams, we can always transform most of them into each other through linear combination. there exists a systematic way of selecting a subset of vectors that characterize all roots we call these the simple vectors (simple roots).

they are by definition positive and the size of their set is the dimension of the carton sub algebra : r

the only possible angles between these is: 90, 120, 135, 150.

we follow the following rules when drawing a dynkin diagram of simple roots

- simple root is represented by O, if short colour the circle
- 90 degrees between, dont connect. 120, 135, 150 are 1 line, 2 line, 3 line connections respectively
- no loops

- no circle can have 3< lines emanating from it
- removing a circle + rewiring → valid diagram } quite powerful
- if there is a 2/3 line connection between 2 circles all circles on one side are short, and on the other side they are long.
- cutting a line gives diagrams of 2 sub-algebras

see p 50 for some (actually all) diagrams

Casimir operators

we define a casimir operator as one which commutes with all the elements of the algebra. we can always define such an operator firstly, define the killing metric:

$$g_{\mu\nu} = f^{\rho}_{\mu\alpha} \cdot f^{\alpha}_{\nu\rho} \qquad \text{(implied summation)}$$

we define its inverse as $g^{\mu\nu}$, and can then construct the casimir oper.

$$\boxed{C = g^{\mu\nu} X_\mu X_\nu} \qquad \text{(again, summation implied)}$$

which can be proven that it commutes. (proof, see p 52 though won't be asked)

9.3.2: projective representations and lie algebras

the equivalent of a projective repr. we will discuss at the level of the lie algebra. for a phase ϕ we introduce a new generator

$$X_\phi = \sum_i \frac{\partial f_i(x,a,\phi)}{\partial \phi} \cdot \frac{\partial}{\partial x_i}$$

which commutes with everything (the other generators) but results in non-trivial structure factors f^{ϕ}_{ab}.

the specific form of these f^{ϕ}_{ab} is specified through the jacobi identity there is an example of this being done on p52