Def^z

28/02 Hfdst 2

$\Rightarrow \exists$ lokaal e unieke opl v $f(x,y)=0$ en (x_0, y_0)
als $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$
Bepaal e vgl vh raakvlak en $(x_0, y(x_0))$

p34 ③ a) $xy - \sin y$ (y ifv x)

① Stelling vd impliciete fct

$f(x,y) = xy - \sin y$

$\Rightarrow \exists$ lokaal e unieke opl en (x_0, y_0) als

$x_0 y_0 - \sin y_0 = 0$ en $x_0 - \cos y_0 \neq 0$

② Vgl vr $y'(x)$ afleiden

$xy(x) - \sin y(x) = 0 \quad \forall x$

afl^z vr x

$\Leftrightarrow y(x) + xy'(x) - \cos y(x) \cdot y'(x) = 0 \quad \forall x$

$\Leftrightarrow y'(x) = \dfrac{-y(x)}{x - \cos y(x)}$ $\neq 0$ vorkomt hier $\log$

③ Vgl v raakvlak en $(x_0, y(x_0))$

raakvlak = rechte

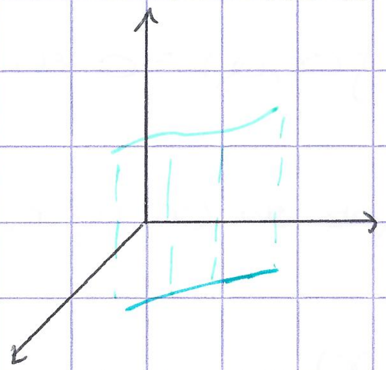
$\Rightarrow y = \underbrace{y'(x_0)}_{\text{rico}}(x - x_0) + y(x_0)$

zorgen bij $x = x_0$
dat $y = y(x_0)$

2

Hfdst 3

$\Rightarrow$ Over scalair veld



Hier $\int_\Gamma f\,ds$

① Bepaal par-vrst. $\vec{\varphi}$ vr Γ

* $\vec{\varphi}[a,b] = \Gamma$ $\quad (\vec{\varphi} : [a,b] \to \Gamma)$
  $\hookrightarrow$ bijectie
* $\vec{\varphi}$ glad (C^1)
* $\vec{\varphi}'(t) \neq \vec{0} \quad \forall t \in]a,b[$

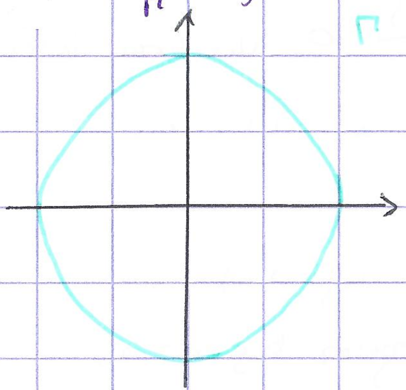
② Bereken $\|\vec{\varphi}'(t)\|$

③ $\int_\Gamma f\,ds = \int_a^b f(\vec{\varphi}(t)) \cdot \|\vec{\varphi}'(t)\|\,dt$

$\Rightarrow$ Vectorveel

③ $\int_\Gamma \vec{f}\,d\vec{s} = \int_a^b \vec{f}(\vec{\varphi}(t)) \cdot \vec{\varphi}'(t)\,dt$

p45 ① a) $\int_\Gamma y^2\,ds$ met Γ eenheidscirkel



① $\vec{\varphi} : [0,2\pi] \to \Gamma : t \to (\cos t, \sin t)$

$\Leftrightarrow \vec{\varphi}(t) = (\cos t, \sin t) \quad 0 \leq t \leq 2\pi$

② $\|\vec{\varphi}'(t)\| = \|(-\sin t, \cos t)\|$
$= \sqrt{(-\sin t)^2 + (\cos t)^2}$
$= 1$

③ $\int_\Gamma y^2\,ds = \int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt = \pi$

b) $\int_\Gamma z\,ds$

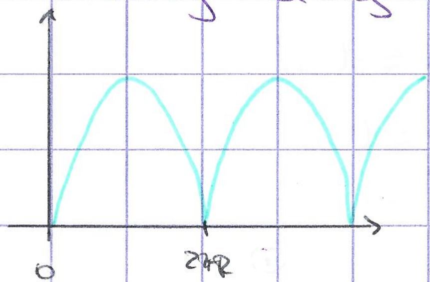
① $\vec{\varphi}(t) = (t\cos t, t\sin t, t) \quad 0 \leq t \leq a$

② $\|\vec{\varphi}'(t)\| = \|(\cos t - t\sin t, \sin t + t\cos t, 1)\| \quad (\neq 0)$
$= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}$
$= \sqrt{2+t^2}$

③ $\int_\Gamma z\,ds = \int_0^a t\sqrt{2+t^2}\,dt = \left[\frac{(2+t^2)^{\frac{3}{2}}}{3/2 \cdot 2}\right]_0^a = \frac{(a^2+2)^{3/2}}{3} - \frac{2\sqrt{2}}{3}$

01/03 d) $\int_\Gamma 1\, ds$ (3)

Γ: boog vd cycloïde $x = R(t - \sin t)$, $y = R(1-\cos t)$



① $\vec{\varphi}(t) = (R(t-\sin t), R(1-\cos t))$

② $\|\vec{\varphi}\,'(t)\| = \|(R(1-\cos t), R(\sin t))\|$

$= \sqrt{R^2(1-\cos t)^2 + R^2(\sin t)^2}$

$= R\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$

$= R\sqrt{2-2\cos t}$

$= 2R\,|\sin \frac{t}{2}|$

$\neq 0$ vr $t \in]0, 2\pi[$

$\sin^2 x = \frac{1-\cos 2x}{2}$

③ $\int_\Gamma 1\, ds = \int_0^{2\pi} 2R\,|\sin(\frac{t}{2})|\, dt$

sin vs 0-2π steeds ⊕

$= \int_0^{2\pi} 2R \sin(\frac{t}{2})\, dt$ $\quad u = \frac{t}{2} \quad du = \frac{1}{2}dt$

$= 2\cdot 2R \int_0^{2\pi} \sin u \, du$

$= 4R\left[-\cos\frac{t}{2}\right]_0^{2\pi}$

$= 4R(-\cos\pi + \cos 0)$

$= 4R(1+1)$

$= 8R$

② a) $\int_\Gamma \vec{f}\cdot d\vec{s}$ met $\vec{f}(x,y,z) = (-y, x, -z/2)$

en Γ: wenteling vd schroeflijn

$x = \cos t \quad y = \sin t \quad z = 2t$

v $(1,0,0)$ tot $(1,0,4\pi)$ $\qquad 0 \leq t \leq 2\pi$

① $\vec{\varphi}(t) = (\cos t, \sin t, 2t)$

② $\vec{\varphi}\,'(t) = (-\sin t, \cos t, 2)$ $\quad (\neq \vec{0})$

③ $\int_\Gamma \vec{f}\cdot d\vec{s}$

$= \int_\Gamma (-y, x, -\frac{z}{2})\cdot d\vec{s}$

$= \int_0^{2\pi} (-\sin t, \cos t, -t)\cdot(-\sin t, \cos t, 2)\, dt$

$= \int_a^b f(\vec{\varphi}(t))\cdot\vec{\varphi}\,'(t)\, dt$

$= \int_0^{2\pi} (\sin^2 t + \cos^2 t - 2t)\, dt$

$= \int_0^{2\pi} (1-2t)\, dt$

$= \int_0^{2\pi} dt - \int_0^{2\pi} 2t\, dt$

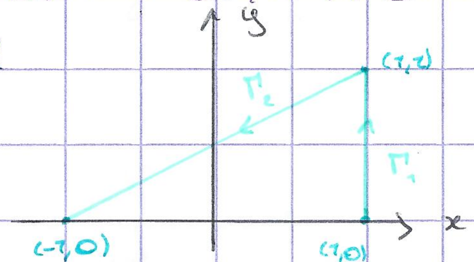
$= [t]_0^{2\pi} - \left[\frac{\cancel{2}t^2}{\cancel{2}}\right]_0^{2\pi}$

$= 2\pi - 0 - 4\pi^2 - 0 = 2\pi - 4\pi^2$

(4)

d) $\int_\Gamma \vec{f}\cdot d\vec{s}$ met $\vec{f}(x,y) = (x-y^3, x^3)$

en Γ d gebroken lijn v $A(1,0)$ via $B(1,1)$ nr $C(-1,0)$



(1) $\vec{\varphi}_1(t) = (1,t) \qquad 0 \le t \le 1$

! $\vec{\varphi}_2(t) = (1-2t, 1-t) \qquad 0 \le t \le 1$

Zin speelt een rol

[nt $\vec{\varphi}_2(v) = (v, \frac{1}{2}v + \frac{1}{2})$ met $-1 \le v \le 1 \Leftrightarrow 0 \le t \le 1$]

nt zelfde grenzen

(2) $\vec{\varphi}_1{}'(t) = (0,1) \qquad (\neq \vec{0})$

$\vec{\varphi}_2{}'(t) = (-2,-1) \qquad (\neq \vec{0})$

(3) $\int_\Gamma (x-y^3, x^3)\cdot d\vec{s}$

$= \int_{\Gamma_1} (x-y^3, x^3)\, d\vec{s} + \int_{\Gamma_2} (x-y^3, x^3)\cdot d\vec{s}$

$= \int_0^1 (1-t^3, 1)\cdot(0,1)\,dt + \int_0^1 (1-2t-(1-t)^3, (1-2t)^3)\cdot(-2,-1)\,dt$

$= \int_0^1 1\,dt + \int_0^1 -2(1-2t-(1-t)^3) - (1-2t)^3\,dt$

$= [t]_0^1 + \int_0^1 -2(1-2t-(1-3t+3t^2-t^3)) - (1-6t+12t^2-8t^3)\,dt$

$= 1 + \int_0^1 -2+4t+2(1-3t+3t^2-t^3) - (1-6t+12t^2-8t^3)\,dt$

$= 1 + \int_0^1 (\cancel{-2} + 4t + \cancel{2} - \cancel{6t} + 6t^2 - 2t^3 - 1 + \cancel{6t} - 12t^2 + 8t^3)\,dt$

$\cancel{= 1 + \int_0^1 (-1 + 4t - 6t^2 - 2t^3)\,dt}$

$\cancel{= 1 + [-t]_0^1 + \left[\frac{4t^2}{2}\right]_0^1 - \left[\frac{2t^3}{3}\right]_0^1 - \left[\frac{12t^4}{42}\right]_0^1}$

$\cancel{= 1 - 1 + 2 - \frac{2}{3} - \frac{1}{2}}$

$= 1 + \int_0^1 (-1 + 4t - 6t^2 + 6t^3)\,dt$

$= 1 + [-t]_0^1 + \left[\frac{4t^2}{2}\right]_0^1 + \left[-\frac{6t^3}{3}\right]_0^1 + \left[\frac{6t^4}{4}\right]_0^1$

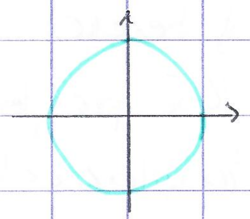
$= 1 - 1 + 2 - 2 + \frac{3}{2}$

$= \frac{3}{2}$

g) $\displaystyle\int_\Gamma \frac{x\,dy - y\,dx}{(x^2+y^2)^{3/2}}$ (5)

met Γ de in tegenwijzerzin doorlopen eenheidscirkel

(1) $\vec{\varphi}(t) = (\cos t, \sin t) \qquad 0 \le t \le 2\pi$

(2) 

$\vec{\varphi}\,'(t) = (-\sin t, \cos t)$

(3) $\displaystyle\int_\Gamma \frac{x\,dy - y\,dx}{(x^2+y^2)^{3/2}}$

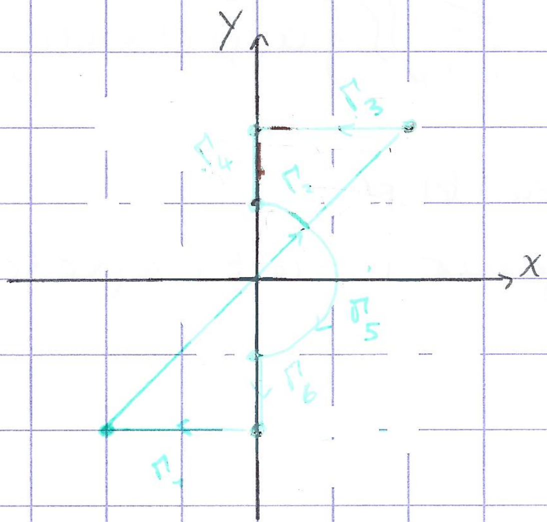
$[x^2+y^2 = \cos^2 t + \sin^2 t = 1]$

$\displaystyle = \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt$

$\displaystyle = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$

$= 2\pi$

~~h)~~ $\displaystyle\oint_\Gamma \left(xy - \frac{y^2}{2}\right)dx$, met Γ als volgt: de gebroken lijn v $(0,-2)$ via $(-2,-2)$, $(2,2)$ en $(0,2)$ nr $(0,1)$, gevolgd dr e halve cirkel met O als middelpt in wijzerzin en een lijnstuk nr h beginpt v Γ.



(1) $\vec{\varphi}_1(t) = (-2t, -2) \qquad (0 \le t \le 1)$

$\vec{\varphi}_1{}'(t) = (-2, 0)$

$\vec{\varphi}_2(t) = (-2+4t, -2+4t) \to {}' = (4,4)$

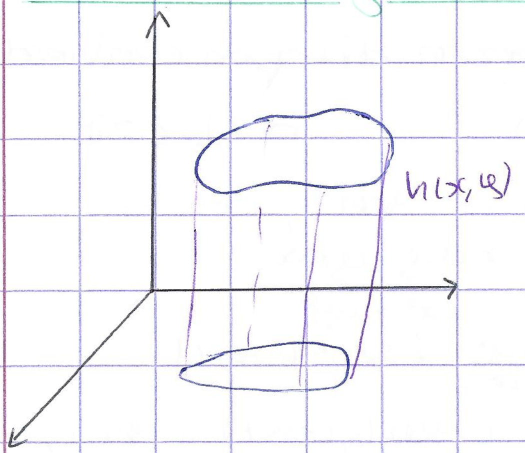
$\vec{\varphi}_3(t) = (2-2t, 2) \to {}' = (-2, 0)$

$\vec{\varphi}_4(t) = (0, 2-t) \to {}' = (0,-1)$

$\vec{\varphi}_5(t) = ($

Hfdst 4

Dubbelintegralen



1) Teken gebied K

2) Bepaal of K x-of y-projbr is (of beide)

→ Stel K x-projbr

$K = \{(x,y) \mid a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}$

3) Bereken integraal

$$\iint_K f(x,y)\,dx\,dy = \int_a^b dx \int_{f_1(x)}^{f_2(x)} f(x,y)\,dy$$

→ Als K niet x-of y-projbr is

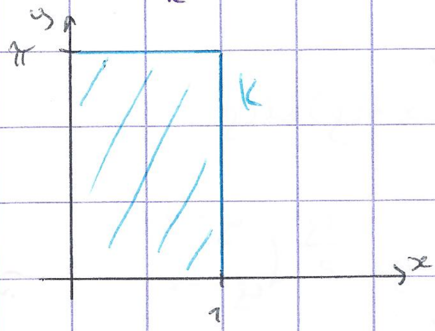
⇒ Opl: Split K in gebied $K_1, \ldots, K_n$ die wel x- of y-projbr zijn

$$\iint_K f(x,y)\,dx\,dy = \sum_{i=1}^{n} \iint_{K_i} f(x,y)\,dx\,dy$$

→ Als integr. te ingewikkeld is

⇒ Voer transformatie uit (bv poolcoörd)

① b) $\iint_K y\cos(xy)\,dx\,dy$ met $K = \{(x,y) \| 0 \le x \le 1, 0 \le y \le \pi\}$ ⑦



K: y-projbr

$$\iint_K y\cos(xy)\,dy\,dx = \int_0^{\pi} dy \int_0^1 y\cos(xy)\,dx$$

$$= \int_0^{\pi} y\,dy \left[\frac{\sin(xy)}{y}\right]_{x=0}^{x=1}$$

$$= \int_0^{\pi} \frac{\cancel{y}\sin y}{\cancel{y}}\,dy$$

$$= [-\cos y]_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$= 1+1$$

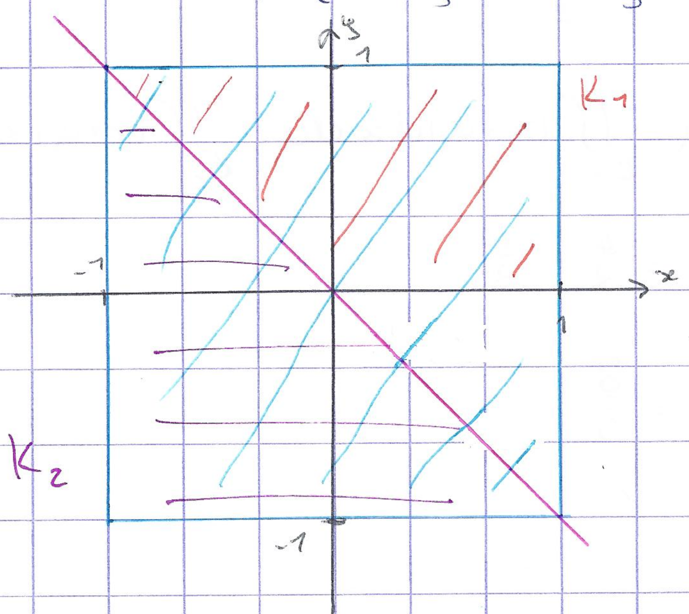
$$= 2$$

c) $\iint_K |x+y|\,dx\,dy$ met $K = \{(x,y) \| |x| \le 1, |y| \le 1\}$

! Bij Abs^e waarde ⇒ opsplitsen $\hookrightarrow -1 \le x \le 1, -1 \le y \le 1$

$$= \iint_{K_1} (x+y)\,dx\,dy + \iint_{K_2} -(x+y)\,dx\,dy$$

$K_1 = \{(x+y) \in K \mid x+y \ge 0\}$ $K_2 = \{(x,y) \in K \mid x+y \le 0$



K_1 x-projbr

8

$K_1 = \{(x,y) \mid -1 \le x \le 1 \mid -x \le y \le 1\}$

$$\iint_{K_1} (x+y)\,dx\,dy = \int_{-1}^{1} dx \int_{-x}^{1} (x+y)\,dy$$

$$= \int_{-1}^{1} dx \left[xy + \frac{y^2}{2}\right]_{y=-x}^{y=1}$$

$$= \int_{-1}^{1} \left(x + \frac{1}{2} + x^2 - \frac{x^2}{2}\right) dx$$

$$= \int_{-1}^{1} \left(\frac{1}{2} + x + \frac{1}{2}x^2\right) dx$$

$$= \left[\frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{6}\right]_{-1}^{1}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{6} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{6}$$

$$= \frac{4}{3}$$

$K_2 = \{(x,y) \mid -1 \le x \le 1, -1 \le y \le -x\}$

$$\iint_{K_2} -(x+y)\,dx\,dy = \int_{-1}^{1} dx \int_{-1}^{-x} -(x+y)\,dy$$

$$= \int_{-1}^{1} dx \left[-xy - \frac{y^2}{2}\right]_{y=-1}^{y=-x}$$

$$= \int_{-1}^{1} \left(x^2 - \frac{x^2}{2} - x + \frac{1}{2}\right) dx$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx$$

$$= \left[\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right]_{-1}^{1}$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} + \frac{1}{6}$$

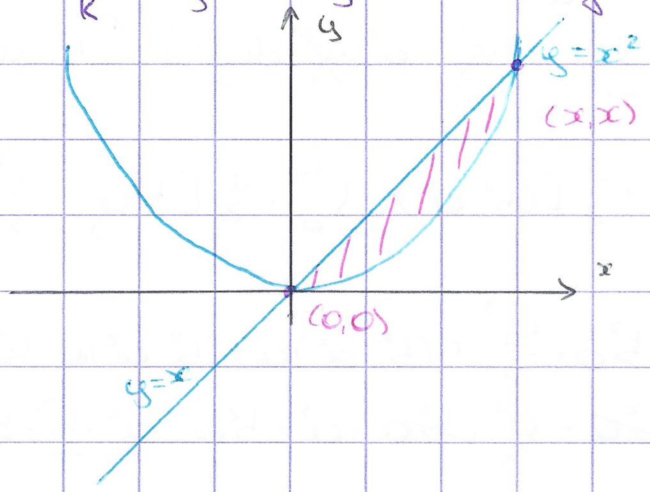
$$= \frac{4}{3}$$

$$\Rightarrow \iint_{K} |x+y|\,dx\,dy = \iint_{K_1} |x+y|\,dx\,dy + \iint_{K_2} |x+y|\,dx\,dy$$

$$= \frac{4}{3} + \frac{4}{3}$$

$$= \frac{8}{3}$$

e) $\iint_K xy \, dx\, dy$ met K begrensd dr $y = x$ en $y = x^2$



$K = \{(x,y) \mid 0 \leq x \leq 1, x^2 \leq y \leq x\}$

$$\iint_K xy\, dx\, dy = \int_0^1 dx \int_{x^2}^{x} xy\, dy$$

$$= \int_0^1 dx = \left[x \frac{y^2}{2}\right]_{y=x^2}^{y=x}$$

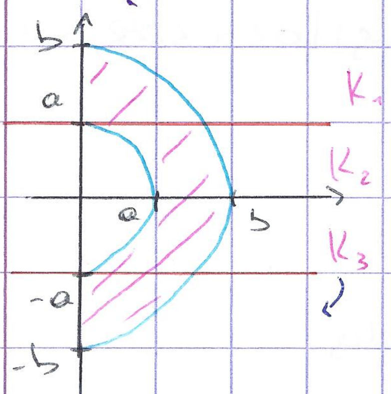
$$= \int_0^1 \left(\frac{x^3}{2} - \frac{x^5}{2}\right) dx$$

$$= \left[\frac{x^4}{8} - \frac{x^6}{12}\right]_{x=0}^{x=1}$$

$$= \frac{1}{8} - \frac{1}{12}$$

$$= \frac{1}{24}$$

h) $\iint_K x\, dx\, dy$ met K h deel vh rechter halfvlak begrepen tss $x^2 + y^2 = b^2$ en $x^2 + y^2 = a^2$ $(0 < a < b)$.



$$\iint_K x\, dx\, dy = \iint_{K_1} x\, dx\, dy + \iint_{K_2} x\, dx\, dy + \iint_{K_3} x\, dx\, dy$$

$K_1 = \{(x,y) \mid a \leq y \leq b, 0 \leq x \leq \sqrt{b^2 - y^2}\}$

$K_2 = \{(x,y) \mid -a \leq y \leq a, \sqrt{a^2-y^2} \leq x \leq \sqrt{b^2-y^2}\}$

$K_3 = \{(x,y) \mid -b \leq y \leq -a, 0 \leq x \leq \sqrt{b^2-y^2}\}$

$$I = \int_a^b dy \int_0^{\sqrt{b^2-y^2}} x\, dx + \int_{-a}^{a} dy \int_{\sqrt{a^2-y^2}}^{\sqrt{b^2-y^2}} x\, dx + \int_{-b}^{-a} dy \int_0^{\sqrt{b^2-y^2}} x\, dx$$

$$= \int_a^b dy \left[\frac{x^2}{2}\right]_0^{\sqrt{b^2-y^2}} + \int_{-a}^{a} dy \left[\frac{x^2}{2}\right]_{\sqrt{a^2-y^2}}^{\sqrt{b^2-y^2}} + \int_{-b}^{-a} dy \left[\frac{x^2}{2}\right]_0^{\sqrt{b^2-y^2}}$$

$$= \int_a^b \frac{(b^2-y^2)}{2} dy + \int_{-a}^{a} \left(\frac{b^2-y^2}{2}\right) - \left(\frac{a^2-y^2}{2}\right) dy + \int_{-b}^{0} \frac{b^2-y^2}{2} dy$$

$$= \frac{1}{2}\int_a^b b^2 - y^2 \, dy + \frac{1}{2}\int_{-a}^{a} b^2 - \cancel{y^2} - a^2 + \cancel{y^2} \, dy + \frac{1}{2}\int_{-b}^{b} b^2 - y^2 \, dy$$

$$= \frac{1}{2}\left[b^2 y - \frac{y^3}{3}\right]_a^b + \frac{1}{2}\left[(b^2 - a^2)y\right]_{-a}^{a} + \frac{1}{2}\left[b^2 y - \frac{y^3}{3}\right]_{-b}^{b}$$

$$= \frac{1}{2}\left(b^2 b - \frac{b^3}{3} - b^2 a + \frac{a^3}{3} + (b^2 - a^2)a + (b^2 - a^2)a + b^2 b - \frac{b^3}{3} + b^2 b - \frac{b^3}{3} \right)$$

$$= \frac{1}{2}\left(b^3 - \frac{b^3}{3} - \cancel{b^2 a} + \frac{a^3}{3} + \cancel{b^2 a} - a^3 + b^2 a - a^3 + b^3 - \frac{b^3}{3} + b^3 - \frac{b^3}{3} \right)$$

$$= \frac{1}{2}\left(\frac{2b^3}{3} + \frac{2b^3}{3} + \frac{2b^3}{3} + b^2 a - 2a^3 \right)$$

$$= \frac{1}{2}\left(2b^3 + b^2 a - 2a^3 \right)$$

$$= b^3 + \frac{1}{2} b^2 a - a^3$$

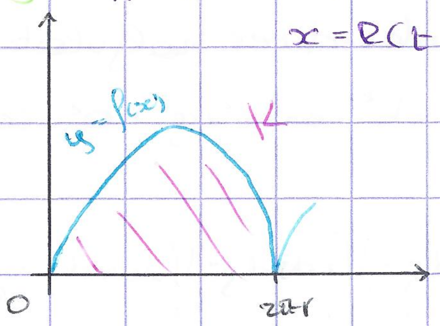
...

$$= \frac{2}{3}(b^3 - a^3)$$

j) $\iint_K y \, dx \, dy$ met K begrensd dr x-as en 1 boog van cycloïde

$x = R(t - \sin t)$, $y = R(1 - \cos t)$

$K = \{(x, y) \mid 0 \leq x \leq 2\pi R, \ 0 \leq y \leq f(x)\}$



$$\iint_K y \, dx \, dy = \int_0^{2\pi R} dx \int_0^{f(x)} y \, dy$$

$$= \int_0^{2\pi R} dx \, \frac{(f(x))^2}{2}$$

$x = R(t - \sin t) \Rightarrow dx = R(1 - \cos t)\, dt$

$y = f(x) = R(1 - \cos t)$

$$= \int_0^{2\pi} R^2 \frac{(1 - \cos t)^2}{2} \cdot R(1 - \cos t)\, dt$$

$$= \frac{R^3}{2}\int_0^{2\pi} (1 - 3\cos t + 3\cos^2 t - \cos^3 t)\, dt$$

$$= \frac{R^3}{2} 2\pi + \frac{3R^3}{2} \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$\cos t$ en $\cos^3 t \rightarrow 2\pi$-per

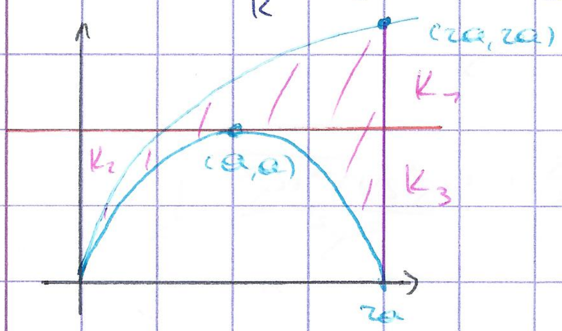
$\rightarrow$ bijdrage $= 0$

$$= R^3 \pi + \frac{3R^3}{2} \cdot \frac{1}{2} 2\pi$$

$$= \frac{5}{2} R^3 \pi$$

(2) b) $\displaystyle\int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\, dy$, met $a > 0$

$$= \iint_K f(x,y)\, dy$$



$K = \{(x,y) \mid 0 \le x \le 2a, \sqrt{2ax-x^2} \le y \le \sqrt{2ax}\}$

Schrijf grenzen als x ifv y

* $y = \sqrt{2ax} \Leftrightarrow y^2 = 2ax \Leftrightarrow x = \frac{1}{2a} y^2$
* $y = \sqrt{2ax - x^2} \Leftrightarrow y^2 = 2ax - x^2$

$\Leftrightarrow x^2 - 2ax + a^2 - a^2 = -y^2$

$\Leftrightarrow x^2 - 2ax + a^2 = a^2 - y^2$

$\Leftrightarrow (x-a)^2 = a^2 - y^2$

$\Leftrightarrow x = a \pm \sqrt{a^2 - y^2}$

$\Rightarrow$ begrenzing $|y| \le a$

$\rightarrow K_1 = \{(x,y) \mid a \le y \le 2a, \frac{y^2}{2a} \le x \le 2a\}$

$\rightarrow K_2 = \{(x,y) \mid 0 \le y \le a, \frac{y^2}{2a} \le x \le a - \sqrt{a^2 - y^2}\}$

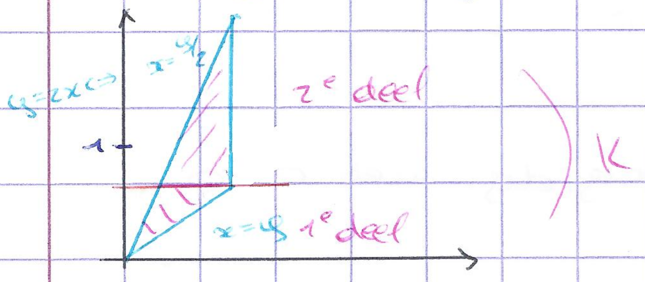
$\rightarrow K_3 = \{(x,y) \mid 0 \le y \le a, a + \sqrt{a^2 - y^2} \le x \le 2a\}$

$$\Rightarrow I = \int_a^{2a} dy \int_{y^2/2a}^{2a} f(x,y)\, dx + \int_0^a dy \int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)\, dx + \int_0^a dy \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\, dx$$

3 $\displaystyle\int_0^1 dy \int_{y/2}^{y} \frac{xy^2}{\sqrt{x^3+y^3}}\,dx + \int_1^2 dy \int_{y/2}^{1} \frac{xy^2}{\sqrt{x^3+y^3}}\,dx$

Idee: integr. volgorde omwisselen

$$= \iint_K \frac{xy^2}{\sqrt{x^3+y^3}}\,dx\,dy$$



$$I = \int_0^1 dx \int_x^{2x} \frac{xy^2}{\sqrt{x^3+y^3}}\,dy$$

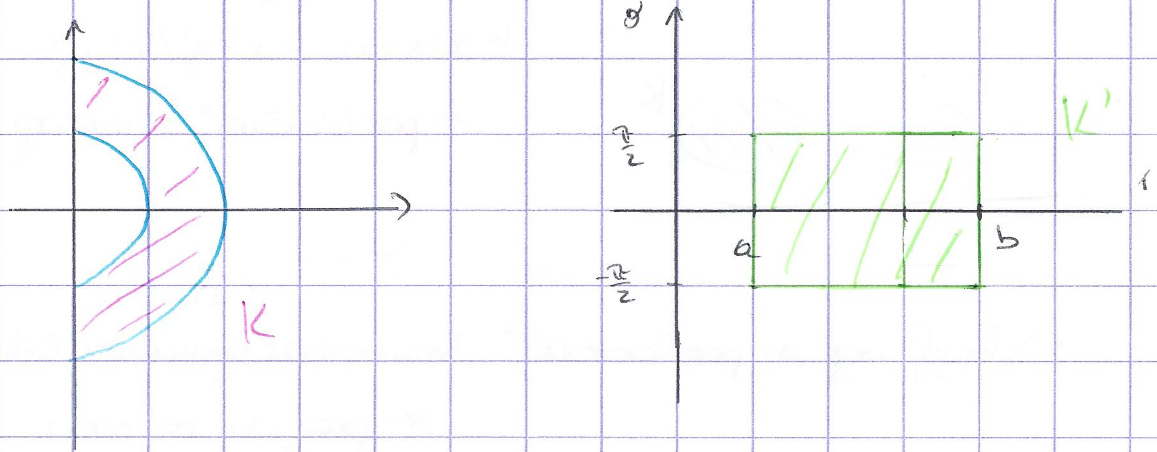
$$= \int_0^1 dx \left[\frac{1}{\sqrt{x^3+y^3}}\cdot\frac{xy^3}{3} + \frac{x\,\cancel{y^2}-2\sqrt{x^3+y^3}}{3y^2} \right]_x^{2x}$$

...

...

$$= \frac{4(3-\sqrt{2})}{21}$$

4 a) $\iint_K x \, dx\,dy$, met K h deel v/h R-halfvlak tss d cirkels met straal a en b > a



$K = \{(r, \theta) \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ a \leq r \leq b\}$

$x = r\cos\theta \qquad y = r\sin\theta$

$\iint_K x \, dx\,dy = \iint_{K'} r\cos\theta \, \boxed{r} \, dr\,d\theta$

$\boxed{r}$ = Jacobiaan (Bij cirkelcoörd: = r)

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_a^b r^2 \cos\theta \, dr$

$= \int_{-\pi/2}^{\pi/2} \cos\theta \left(\frac{b^3 - a^3}{3}\right) d\theta$

$= \frac{2}{3}(b^3 - a^3)$

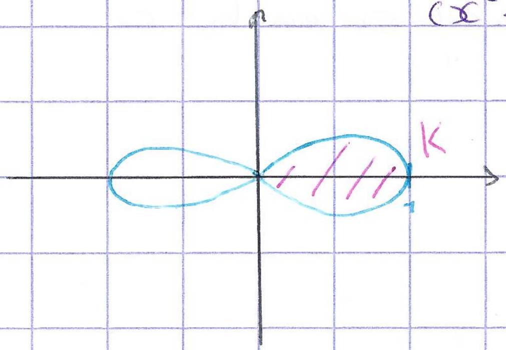
(14) 21/03

d) $\iint_K \frac{dx\,dy}{(1+x^2+y^2)^2}$ met K de rechterhelft v/h lemniscaat

$(x^2+y^2)^2 = x^2-y^2$

↳ ook zien v x^2+y^2 / x^2-y^2

⇒ poolcoörd² te nuttig zijn



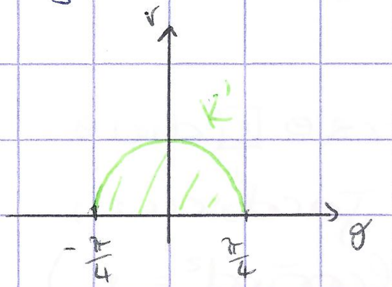
→ Schrijf vgl in poolcoörd² ↳ meestal gemakkelijker om als θ-proj: bv te zien

$x = r\cos\theta$

$y = r\sin\theta$

⇒ In vgl zetten ⇒ $r^4 = r^2\cos 2\theta$

⇔ $r^2 = \cos 2\theta$

→ 

Rechterhelft lemniscaat

⇒ begrenzing v/ K: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

(r altijd pos ⇒ $\cos 2\theta$ moet dus ook altijd pos zijn)

$\Rightarrow K' = \{(\theta, r) \mid -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq \sqrt{\cos 2\theta}\}$

De grens $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ volgt uit h feit dat

$0 \leq \sqrt{\cos 2\theta} \Rightarrow -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

Jac.

$$\rightarrow \iint_K \frac{dx\,dy}{(1+x^2+y^2)^2} = \iint_{K'} \frac{\boxed{r}\,dr\,d\theta}{(1+r^2\cos^2\theta + r^2\sin^2\theta)^2}$$

$$= \iint_{K'} \frac{r\,dr\,d\theta}{(1+r^2)^2}$$

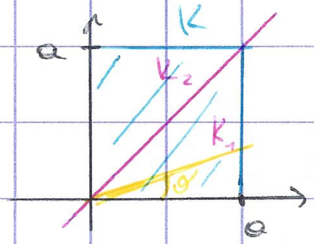
$$= \int_{-\pi/4}^{\pi/4} d\theta \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr$$

$= \ldots$

$= \frac{\pi-2}{4}$

Extra oef.

Bereken opp v vierkant met zijde a in poolcoörd²



$x = r\cos\theta$

$y = r\sin\theta$

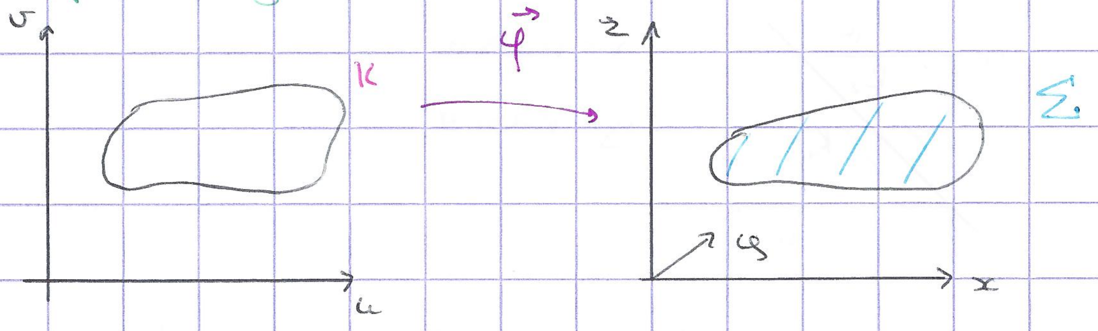
$opp(K_1) = opp(K_2) \Rightarrow$ h volstaat 1 gebied te berekenen

$$\begin{aligned} opp\, K &= \iint_K 1\, dx\, dy \\ &= 2 \iint_{K_1} 1\, dx\, dy \\ &= 2 \int_0^{\pi/4} d\theta \int_0^{a/\cos\theta} \boxed{r}\, dr \\ &= \int_0^{\pi/4} \frac{a^2}{\cos^2\theta}\, d\theta \\ &= a^2 \end{aligned}$$

Jac

Hfdst 5

Opp-integralen



1e Soort: Scalairveld

$$\iint_\Sigma f(x,y,z)\,d\sigma$$

① Zoek PV vr Σ $\qquad \vec{\varphi}(u,v) : K_{\subset \mathbb{R}^2} \rightarrow \Sigma$

② Bereken $\left\| \frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v} \right\|$

③ $\iint_\Sigma f(x,y,z)\,d\sigma = \iint_K f(\vec{\varphi}_1(u,v), \vec{\varphi}_2(u,v), \vec{\varphi}_3(u,v)) \cdot \left\| \frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v} \right\| du\,dv$

2e Soort: Vectorveld

$$\iint_\Sigma \vec{f}(x,y,z) \cdot d\vec{\sigma}$$

① Zoek PV vr Σ $\qquad \vec{\varphi}(u,v) : K_{\subset \mathbb{R}^2} \rightarrow \Sigma$

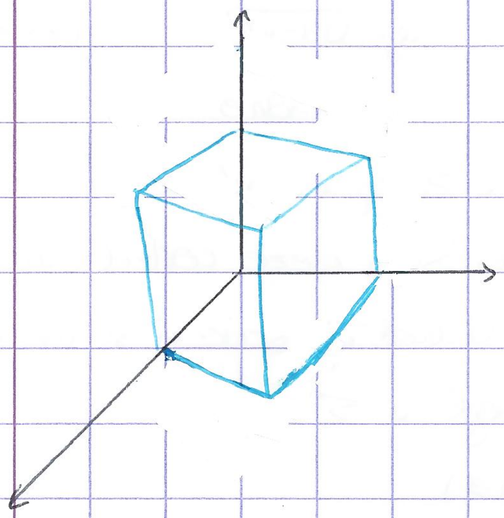
② $\frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v}$ $\qquad$ Zin normaalvector

u & v omwisselen → verand^{e} v zin

③ $\iint_\Sigma \vec{f}(x,y,z) \cdot d\vec{\sigma} = \iint_K \vec{f}(\varphi(u,v)) \cdot \left(\frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v} \right) du\,dv$

1 a) $\iint_\Sigma (x+y)z\, d\sigma$, met Σ h opp v/d kubus $[0,1]^3$.

[x, y → u, v en z laten afhangen v x en y]



$$\iint_\Sigma (x+y)z\,d\sigma = \sum_{i=1}^{6} \iint_{\Sigma_i} (x+y)z\,dz$$

Σ_i zijn de 6 zijvlakken

Σ_1 = bovenvlak

1) PV $\quad \vec{\varphi}(u,v) \to (u, v, 1) \quad$ $0 \leq u \leq 1$, $0 \leq v \leq 1$

2) $\frac{\partial \vec{\varphi}}{\partial u} = (1,0,0) \qquad \frac{\partial \vec{\varphi}}{\partial v} = (0,1,0)$

$$\frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1)$$

$$\Rightarrow \left\| \frac{\partial \vec{\varphi}}{\partial u} \times \frac{\partial \vec{\varphi}}{\partial v} \right\| = 1$$

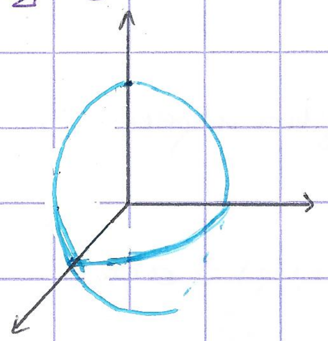
3)
$$\iint_{\Sigma_1} (x+y)z\,d\sigma = \int_0^1 du \int_0^1 (u+v)\,dv = \int_0^1 du \left[uv + \frac{u^2}{2}\right]_0^1 = \int_0^1 \left(u + \frac{1}{2}\right) du = \left[\frac{u^2}{2} + \frac{1}{2}u\right]_0^1 = 1$$

⇒ Som v/d zijvlakken:

$$\iint_\Sigma (x+y)z\,d\sigma = 3$$

Keuze PV: bv linkervlak: $x = 0$ dus geen gelijkstellen $= 0$ en andere coörd^e = par^e

18 22/03 b) $\iint_\Sigma y(z^2+x^2)\,d\sigma$, met Σ geg dr $y=\sqrt{1-x^2-z^2}$



1) PV $x = r\cos\theta$ $(0 \le \theta \le 2\pi)$

$y = \sqrt{1-r^2}$ $(0 \le r \le 1)$

$z = r\sin\theta$

Neem proj. v Σ op xz-vlak.
Parametriseer deze cirkel (met z = vrije par). Vind d par-vrst vr coörd y dr vgl v Σ

2) $\frac{\partial\vec{\varphi}}{\partial r} = \left(\cos\theta, \frac{-r}{\sqrt{1-r^2}}, \sin\theta\right)$

$\frac{\partial\vec{\varphi}}{\partial\theta} = (-r\sin\theta, 0, r\cos\theta)$

$$\Rightarrow \frac{\partial\vec{\varphi}}{\partial r} \times \frac{\partial\vec{\varphi}}{\partial\theta} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \cos\theta & \frac{-r}{\sqrt{1-r^2}} & \sin\theta \\ -r\sin\theta & 0 & r\cos\theta \end{vmatrix}$$

$$= \left(\frac{-r^2\cos\theta}{\sqrt{1-r^2}}, -r, -\frac{r^2\sin\theta}{\sqrt{1-r^2}}\right)$$

$$\Rightarrow \left\| \frac{\partial\vec{\varphi}}{\partial r} \times \frac{\partial\vec{\varphi}}{\partial\theta} \right\| = \sqrt{\frac{r^4\cos^2\theta}{1-r^2} + r^2 + \frac{r^4\sin^2\theta}{1-r^2}}$$

$$= \sqrt{\frac{r^4}{1-r^2} + \frac{r^2(1-r^2)}{1-r^2}}$$

$$= \sqrt{\frac{r^4+r^2-r^4}{1-r^2}}$$

$$= \sqrt{\frac{r^2}{1-r^2}}$$

$$= \frac{r}{\sqrt{1-r^2}}$$

3) $$\iint_\Sigma y(z^2+x^2)\,d\sigma = \int_0^{2\pi} d\theta \int_0^1 \sqrt{1-r^2}\cdot(r^2\cos^2\theta + r^2\sin^2\theta)\cdot\frac{r}{\sqrt{1-r^2}}\,dr$$

$$= \int_0^{2\pi} d\theta \int_0^1 r\,r^2\,dr$$

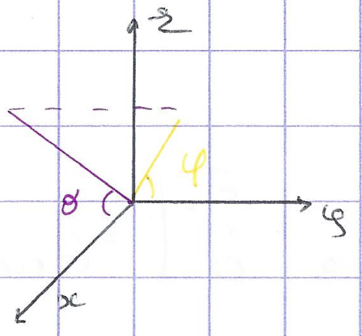
$$= \int_0^{2\pi} d\theta \left[\frac{r^4}{4}\right]_0^1$$

$$= \int_0^{2\pi} \frac{1}{4}\,d\theta$$

$$= \frac{2\pi}{4}$$

$$= \frac{\pi}{2}$$

⟷ Andere PV



1) PV

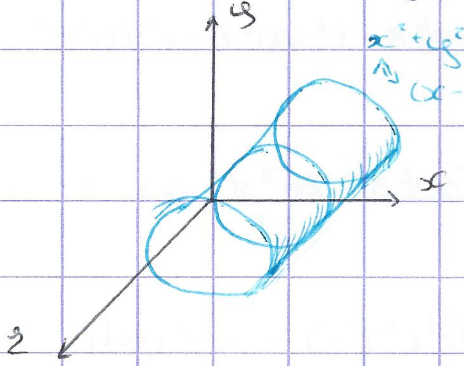
$x = \cos\theta \sin\varphi$ $\quad (0 \le \varphi \le \frac{\pi}{2})$

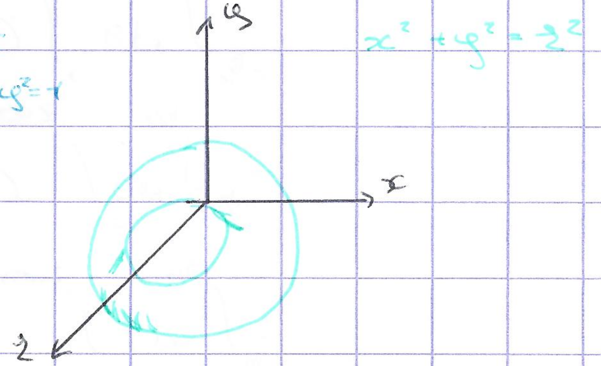
$y = \cos\varphi$ $\quad (0 \le \theta \le 2\pi)$

$z = \sin\theta \sin\varphi$

2) $\left\| \frac{\partial \vec{\varphi}_0}{\partial \theta} \times \frac{\partial \vec{\varphi}_0}{\partial \varphi} \right\| = \sin\varphi$

d) $\iint_{\Sigma} (x^4 - y^4 + y^2 z^2 - z^2 x^2 + 1)\, d\sigma$, met Σ h opp dr d cilinder $x^2 + y^2 = 2x$ uitgesneden uit h bovenste blad vh kegelopp $x^2 + y^2 = z^2$

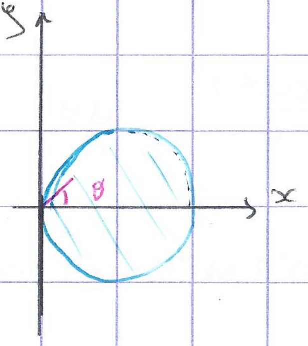




1) PV

1) Eerst opp op vlak (hier xy-vlak) projecteren $\quad x = r\cos\theta$, $y = r\sin\theta$

2) Parametriseer eerst d cirkel



Schrijf $x^2 + y^2 = 2x$ in poolcoörd.:

$r^2 = 2r\cos\theta$

$\Leftrightarrow r = 2\cos\theta$

$\Rightarrow (0 \le r \le 2\cos\theta)$

$\Rightarrow 0 \le \cos\theta$

$\Rightarrow (-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$

3) kegel $z = r$

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2) $\dfrac{\partial\vec{\varphi}}{\partial r} = (\cos\theta, \sin\theta, 1)$

$\dfrac{\partial\vec{\varphi}}{\partial\theta} = (-r\sin\theta, r\cos\theta, 0)$

$\Rightarrow \dfrac{\partial\vec{\varphi}}{\partial r} \times \dfrac{\partial\vec{\varphi}}{\partial\theta} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta, -r\sin\theta, 1)$

$\Rightarrow \left\| \dfrac{\partial\vec{\varphi}}{\partial r} \times \dfrac{\partial\vec{\varphi}}{\partial\theta} \right\| = \sqrt{2}\, r$

3) $\displaystyle\iint_{\Sigma} (x^4 - y^4 + y^2 z^2 - z^2 x^2 + 1)\, d\sigma$

$\displaystyle = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} (r^4\cos^4\theta - r^4\sin^4\theta + r^2\sin^2\theta r^2 - r^2 r^2\cos^2\theta + 1)\sqrt{2}\, r\, dr$

$\displaystyle = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} (r^4(\cos^4\theta - \sin^4\theta + \sin^2\theta - \cos^2\theta) + 1)\sqrt{2}\, r\, dr$

$\displaystyle = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} (\sqrt{2}\, r^5 (-) + \sqrt{2}\, r)\, dr$

$\displaystyle = \int_{-\pi/2}^{\pi/2} d\theta\, \sqrt{2} \left[\frac{r^6}{6}(-) + \frac{r^2}{2} \right]_0^{2\cos\theta}$

$\displaystyle = \int_{-\pi/2}^{\pi/2} \sqrt{2} \left(\frac{(2\cos\theta)^6}{6} (\cos^4\theta - \sin^4\theta + \sin^2\theta - \cos^2\theta) + \frac{4\cos^2\theta}{2} \right) d\theta$

$\displaystyle = \sqrt{2} \int_{-\pi/2}^{\pi/2} \left(\frac{64\cos^6\theta}{6} (-) + 2\cos^2\theta \right) d\theta$

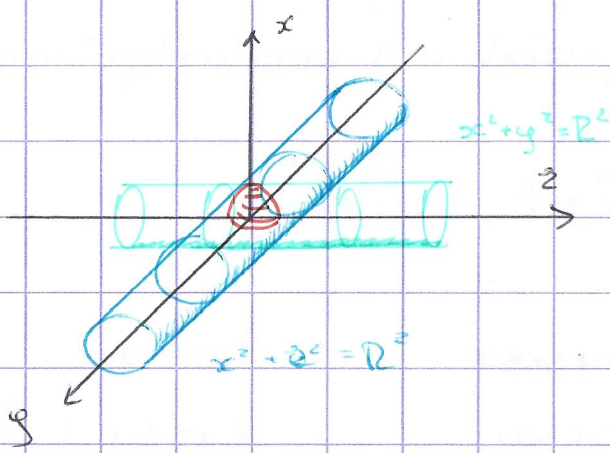
$\displaystyle = \sqrt{2} \int_{-\pi/2}^{\pi/2} \left(\frac{32}{3} \cdot (\cos^{10}\theta - \cos^6\theta\sin^4\theta + \cos^6\theta\sin^2\theta - \cos^8\theta) + 2\cos^2\theta \right) d\theta$

$= \sqrt{2}\, \left[\frac{32}{3} \cdot \left(\frac{\cos^{11}\theta}{11} - \right.\right.$

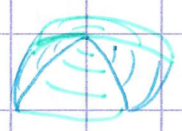
...

$= \pi\sqrt{2}$

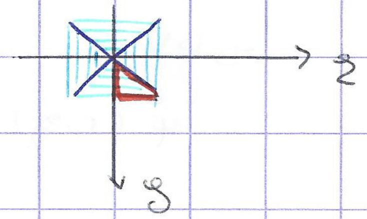
② a) Opp begrensd door d cilinders $x^2+y^2=R^2$ en $x^2+z^2=R^2$



Doorsnede cilinders:



Bovenaanzicht:



Bereken opp vh oranje deel

1) PV $x=\sqrt{R^2-u^2}$

$y=u \qquad (0\le u\le R)$

$z=v \qquad (0\le v\le u)$

2) $\frac{\partial\vec\varphi}{\partial u}=\left(\frac{-u}{\sqrt{R^2-u^2}},1,0\right) \qquad \frac{\partial\vec\varphi}{\partial v}=(0,0,1)$

$\to \frac{\partial\vec\varphi}{\partial u}\times\frac{\partial\vec\varphi}{\partial v}=\begin{vmatrix}\vec e_1 & \vec e_2 & \vec e_3\\ -\frac{u}{\sqrt{R^2-u^2}} & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=\left(1,(-1)^{2+1}\cdot-\frac{u}{\sqrt{R^2-u^2}},0\right)$

$\to \left\|\frac{\partial\vec\varphi}{\partial u}\times\frac{\partial\vec\varphi}{\partial v}\right\|=\sqrt{1+\frac{u^2}{R^2-u^2}}=\sqrt{\frac{R^2-u^2}{R^2-u^2}+\frac{u^2}{R^2-u^2}}=\sqrt{\frac{R^2}{R^2-u^2}}=\frac{R}{\sqrt{R^2-u^2}}$

3) Opp oranje deel

$$\iint_{\Sigma_{oranje}} 1\,d\sigma=\int_0^R du\int_0^u\frac{R}{\sqrt{R^2-u^2}}\,dv$$

$$=\int_0^R\frac{Ru}{\sqrt{R^2-u^2}}\,du$$

...

$=R^2$

Tot^e opp $\Rightarrow \cdot 16 \Rightarrow 16R^2$

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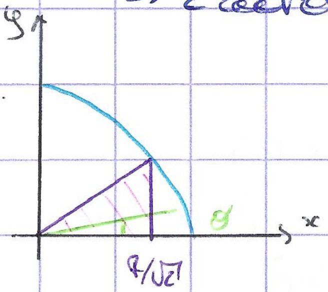
23/03 ② c) Deel vd halve sfeer $x^2+y^2+z^2=R^2$ $(z\geq 0)$ wiv d proj op h xy-vlak d Δ met hoekpt^n $(0,0)$, $(\frac{R}{\sqrt{2}},0)$, $(R/\sqrt{2}, R/\sqrt{2})$

$$\left[\int_0^{\pi/4} \sqrt{2-\frac{1}{\cos^2 t}}\,dt = \frac{\pi}{2}(\sqrt{2}-1)\right]$$

→ PV

$$\vec{\varphi}(r,\theta) \begin{cases} x = r\cos\theta & (0\leq\theta\leq\frac{\pi}{4}) \\ y = r\sin\theta & (0\leq r\leq \frac{R}{\sqrt{2}\cos\theta}) \\ z = \sqrt{R^2-r^2} \end{cases}$$

$\Rightarrow z$ coörd : $r^2+z^2=R^2 \Rightarrow z=\sqrt{R^2-r^2}$ $(z\geq 0)$



$$\frac{\partial\vec{\varphi}}{\partial r} = \left(\cos\theta, \sin\theta, -\frac{r}{\sqrt{R^2-r^2}}\right)$$

$$\frac{\partial\vec{\varphi}}{\partial\theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\Rightarrow \frac{\partial\vec{\varphi}}{\partial r}\times\frac{\partial\vec{\varphi}}{\partial\theta} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \cos\theta & \sin\theta & -\frac{r}{\sqrt{R^2-r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(\frac{r^2\cos\theta}{\sqrt{R^2-r^2}}, \frac{r^2\sin\theta}{\sqrt{R^2-r^2}}, r\right)$$

$$\Rightarrow \left\|\frac{\partial\vec{\varphi}}{\partial r}\times\frac{\partial\vec{\varphi}}{\partial\theta}\right\| = \sqrt{\frac{r^4\cos^2\theta}{R^2-r^2}+\frac{r^4\sin^2\theta}{R^2-r^2}+r^2} = \frac{Rr}{\sqrt{R^2-r^2}}$$

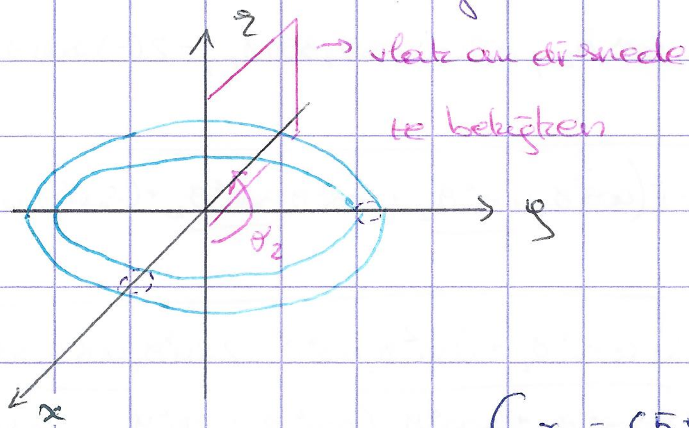
$$\Rightarrow \text{opp} = \iint_\Sigma 1\,d\sigma$$

$$= \int_0^{\pi/4} d\theta \int_0^{R/\sqrt{2}\cos\theta} \frac{Rr}{\sqrt{R^2-r^2}}\,dr$$

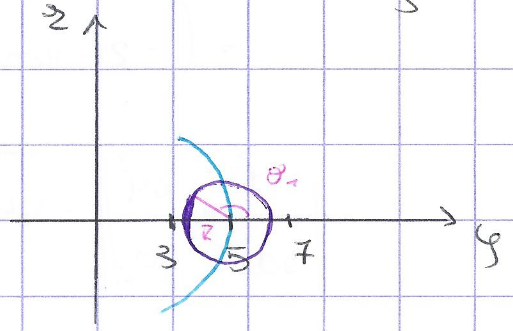
$$=$$

$$= \frac{R^2\pi}{4}(\sqrt{2}-1).$$

b) Opp vh torusopp. ontstaan dr d cirkel met straal 2 te wentelen rond d z-as op afst 5 vh middelpt vd cirkel, h deel dat binnen d sfeer $x^2+y^2+z^2=R^2$ $(3<R<7)$ ligt.



Dsnede met yz-vlak



→ 1) PV $\vec{\varphi}(\theta_1,\theta_2) = \begin{cases} x = (5+2\cos\theta_1)\cos\theta_2 \\ y = (5+2\cos\theta_1)\sin\theta_2 \\ z = 2\sin\theta_1 \end{cases}$

(Parametrisatie vd cirkel → $y = 5+2\cos\theta_1$
$z = 2\sin\theta_1$)

→ $(0 \le \theta_2 \le 2\pi)$

→ Grens vr θ_1 (analytisch)

$x^2+y^2+z^2 \le R^2$ (deel binnen sfeer)

vr welke θ_1 en θ_2 voldaan?

$\Rightarrow (5+2\cos\theta_1)^2\cos^2\theta_2 + (5+2\cos\theta_1)^2\sin^2\theta_2 + 4\sin^2\theta_1 \le R^2$

$25 + 20\cos\theta_1 + 4 \le R^2$

$\cos\theta_1 \le \dfrac{R^2-29}{20}$

$\Rightarrow \left(\mathrm{Bgcos}\left(\frac{R^2-29}{20}\right) \le \theta_1 \le 2\pi - \mathrm{Bgcos}\left(\frac{R^2-29}{20}\right)\right)$

2) $\dfrac{\partial\vec{\varphi}}{\partial\theta_1} = (-2\sin\theta_1\cos\theta_2, -2\sin\theta_1\sin\theta_2, 2\cos\theta_1)$

$\dfrac{\partial\vec{\varphi}}{\partial\theta_2} = (-(5+2\cos\theta_1)\sin\theta_2, (5+2\cos\theta_1)\cos\theta_2, 0)$

$\dfrac{\partial\vec{\varphi}}{\partial\theta_1} \times \dfrac{\partial\vec{\varphi}}{\partial\theta_2} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -2\sin\theta_1\cos\theta_2 & -2\sin\theta_1\sin\theta_2 & 2\cos\theta_1 \\ -(5+2\cos\theta_1)\sin\theta_2 & (5+2\cos\theta_1)\cos\theta_2 & 0 \end{vmatrix}$

$= (-2\cos\theta_1(5+2\cos\theta_1)\cos\theta_2,\ 2(5+2\cos\theta_1)\cos\theta_1\sin\theta_2,$
$-2(5+2\cos\theta_1)\cos^2\theta_2\sin\theta_1 + 2(5+2\cos\theta_1)\sin^2\theta_2\sin\theta_1 \ldots$

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$$\frac{\partial \varphi}{\partial \theta_1} \times \frac{\partial \varphi}{\partial \theta_2} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -2\operatorname{sen}\theta_1\cos\theta_2 & -2\operatorname{sen}\theta_1\operatorname{sen}\theta_2 & 2\cos\theta_1 \\ -(-)\operatorname{sen}\theta_2 & (-)\cos\theta_2 & 0 \end{vmatrix}$$

$$= \left(-2(-)\cos\theta_1\cos\theta_2 ,\ -2(-)\cos\theta_1\operatorname{sen}\theta_2 ,\ -2(-)(\operatorname{sen}\theta_1\cos^2\theta_2 - \operatorname{sen}\theta_1\operatorname{sen}^2\theta_2) \right)$$

$$\left\| \frac{\partial \vec{\varphi}}{\partial \theta_1} \times \frac{\partial \vec{\varphi}}{\partial \theta_2} \right\| = \sqrt{4(-)^2(\cos^2\theta_1\cos^2\theta_2 + \cos^2\theta_1\operatorname{sen}^2\theta_2 + (\operatorname{sen}\theta_1\cos^2\theta_2 - \operatorname{sen}\theta_1\operatorname{sen}^2\theta_2)^2)}$$

$$= 2(-)\sqrt{\cos^2\theta_1 + \operatorname{sen}^2\theta_1\cos^4\theta_2 - 2\operatorname{sen}^2\theta_1\cos^2\theta_2\operatorname{sen}^2\theta_2 + \operatorname{sen}^2\theta_1\operatorname{sen}^4\theta_2}$$

$$= 2(-)\sqrt{\cos^2\theta_1 + \operatorname{sen}^2\theta_1(\cos^4\theta_2 + \operatorname{sen}^4\theta_2 - 2\cos^2\theta_2\operatorname{sen}^2\theta_2)}$$

$$= 2(-) \cdot 1 ?$$

$$= 2(5 + 2\cos\theta_1)$$

3) $\text{app} = \iint_\Sigma 1\, d\sigma$

$$= \int_0^{2\pi} d\theta_2 \int_{\arccos\left(\frac{R^2-29}{20}\right)}^{2\pi - \arccos\left(\frac{R^2-29}{20}\right)} 2(5 + 2\cos\theta_1)\, d\theta_1$$

$$= \int_0^{2\pi} d\theta_2\ 2 \cdot \int_a^b 5 + 2\cos\theta_1\, d\theta_1$$

$$= \int_0^{2\pi} d\theta_2 \cdot \left(2[5\theta_1]_a^b + 4[\operatorname{sen}\theta_1]_a^b \right)$$

$$= \int_0^{2\pi} d\theta_2 \left(10\left(2\pi - \arccos\left(\frac{R^2-29}{20}\right) - \arccos\left(\frac{R^2-29}{20}\right)\right) + 4\left(\operatorname{sen}\left(2\pi - \arccos\left(\frac{R^2-29}{20}\right)\right) - \operatorname{sen}\left(\arccos\left(\frac{R^2-29}{20}\right)\right)\right) \right)$$

$$= \int_0^{2\pi} \left(20\pi - 20\arccos\left(\frac{R^2-29}{20}\right) + 4\left(-2\operatorname{sen}\left(\arccos\left(\frac{R^2-29}{20}\right)\right)\right) \right) d\theta_2$$

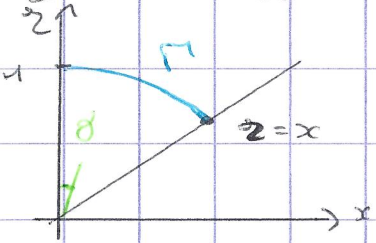
$$= 40\pi\arccos\left(\frac{29-R^2}{20}\right) - 16\pi\operatorname{sen}\left(\arccos\left(\frac{29-R^2}{20}\right)\right)$$

4 e) $\iint_\Sigma \vec{F}\cdot d\vec{\sigma}$ met

$\vec{F}(x,y,z) = (x,y,0)$ en

Σ h gedeelte vd sfeer $x^2+y^2+z^2=1$ dat boven $z=\sqrt{x^2+y^2}$ gelegen is.

Dr snede xz-vlak



1) PV Σ krijg je dr Γ rond z-as te wentelen.

$$\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases} \rightarrow \Gamma \text{ param. vr } \Gamma: \begin{array}{l} x = \sin\theta \\ z = \cos\theta \end{array}$$

We gebruiken φ om rond z-as te wentelen.

$(0 \leq \varphi < 2\pi) \qquad (0 \leq \theta \leq \frac{\pi}{4})$

2) $\frac{\partial \vec{\varphi}_0}{\partial \varphi} \times \frac{\partial \vec{\varphi}_0}{\partial \theta} = (-\sin^2\theta\cos\varphi, -\sin^2\theta\sin\varphi, -\sin\theta\cos\theta)$.

Normaal ⇒ wijst v oorsprong weg.

⇒ vector wijst nt in juiste zin

(kijk nr z-coörd^z en grenzen hoeken. → $z = \cos\theta$, $\theta = 0 \to 1$, $\theta = \frac{\pi}{4} \to \frac{\sqrt{2}}{2}$

v oorspr weg!

3) $\iint_\Sigma (x,y,0)\cdot d\vec{\sigma} = \int_0^{2\pi} d\varphi \int_0^{\pi/4} (\sin$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/4} (\sin\theta\cos\varphi, \sin\theta\sin\varphi, 0)\cdot(\sin^2\theta\cos\varphi, \sin^2\theta\sin\varphi, \sin\theta\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/4} (\sin^3\theta\cos^2\varphi + \sin^3\theta\sin^2\varphi)\, d\theta$$

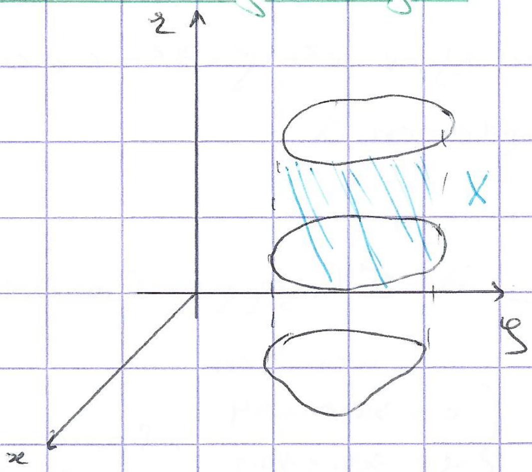
$$= \int_0^{2\pi} d\varphi \int_0^{\pi/4} \sin^3\theta\, d\theta$$

$$= 2\pi\left(\frac{2}{3} - \frac{5\sqrt{2}}{12}\right)$$

Hfdst 6

27/03

Drievoudige integr²



$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$

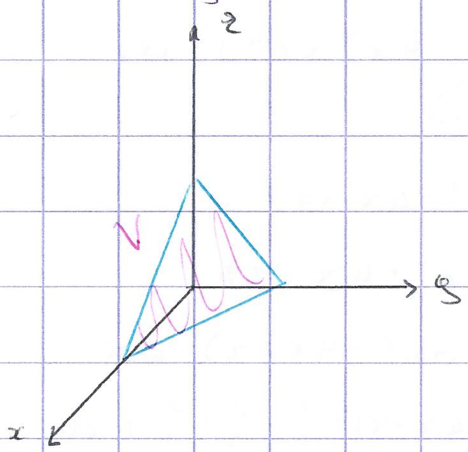
- Stel V is xy-projecteerbaar, i.e.
  $V = \{(x,y,z) \mid (x,y) \in D,\ f_1(x,y) \leq z \leq f_2(x,y)\}$
  vr 'geschikte' fcts f_1 en f_2.

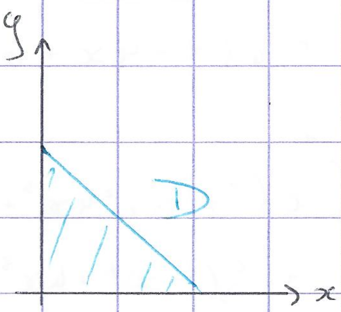
  $$\Rightarrow \iiint_V f(x,y,z)\,dx\,dy\,dz = \iint_D dx\,dy \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz.$$

- Als V niet xy- of xz- of yz-projecteerbaar is:
  Splits in gebieden die dat wel zijn.

- Ev. kan ook e transform nuttig zijn.
  (Zie paragraaf 6.3)

Bereken $\iiint_V xyz\,dx\,dy\,dz$ met V begrensd door $x=0, y=0, z=0,$
$x+y+z=1$





V is xy-projctbr

$$V=\{(x,y,z)\mid (x,y)\in D;\ 0\le z\le 1-x-y\}$$

met $D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1-x\}$

of

$$V=\{(x,y,z)\mid 0\le x\le 1,\ 0\le y\le 1-x,\ 0\le z\le 1-x-y\}$$

grenzen mogen afhangen v par^z ervóór

$$\rightarrow \iiint_V xyz\,dx\,dy\,dz$$
$$=\iint_D xy\,dx\,dy\int_0^{1-x-y} z\,dz$$
$$=\int_0^1 x\,dx\int_0^{1-x} y\,dy\int_0^{1-x-y} z\,dz$$
$$=\int_0^1 x\,dx\int_0^{1-x} y\,dy\left[\frac{z^2}{2}\right]_0^{1-x-y}$$
$$=\int_0^1 x\,dx\,\frac{1}{2}\int_0^{1-x} y\cdot(1-x-y)^2\,dy$$
$$=\frac{1}{2}\int_0^1 x\,dx\int_0^{1-x} y\cdot\left(1+x^2+y^2-2x-2y+2xy\right)dy$$
$$=\frac{1}{2}\int_0^1 x\,dx\int_0^{1-x}(1+x^2-2x)y+2(x-1)y^2+y^3\,dy$$
$$=\frac{1}{2}\int_0^1 x\,dx\left[\frac{1}{2}(1+x^2-2x)y^2+\frac{2}{3}(x-1)y^3+\frac{1}{4}y^4\right]_0^{1-x}$$
$$=\frac{1}{2}\int_0^1 x\cdot\left(\frac{1}{2}(1+x^2-2x)(1-x)^2+\frac{2}{3}(x-1)(1-x)^3+\frac{1}{4}(1-x)^4\,dx\right.$$
$$=\frac{1}{2}\int_0^1 x\left(\frac{1}{2}(1+x^2-2x)(x^2-2x+1)+\frac{2}{3}(x-1)(1-3x+3x^2-x^3)\right.$$
$$\left.+\frac{1}{4}(1-4x+6x^2-4x^3+x^4)\right)dx$$

$$= \frac{1}{2}\int_0^1 x\Big(\frac{1}{2}\big(x^2 - 2x + 1 + x^4 - 2x^3 + x^2 - 2x^3 + 4x^2 - 2x\big)$$

$$+ \frac{2}{3}\big(x - 3x^2 + 3x^3 - x^4 - 1 + 3x - 3x^2 + x^3\big)$$

$$+ \frac{1}{4}\big(1 - 4x + 6x^2 - 4x^3 + x^4\big)\Big)\,dx$$

$$= \frac{1}{2}\int_0^1 x\Big(\frac{1}{2}\big(x^4 - 4x^3 + 6x^2 - 4x + 1\big)$$

$$+ \frac{2}{3}\big(-x^4 + 4x^3 - 6x^2 + 4x - 1\big)$$

$$+ \frac{1}{4}\big(x^4 - 4x^3 + 6x^2 - 4x + 1\big)\,dx$$

$$= \frac{1}{2}\int_0^1 x\,(x^4 - 4x^3 + 6x^2 - 4x + 1)\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)dx$$

$$= \frac{1}{24}\int_0^1 (x^5 - 4x^4 + 6x^3 - 4x^2 + x)\,dx$$

$$= \frac{1}{24}\left[\frac{x^6}{6} - \frac{4}{5}x^5 + \frac{\cancel{6}^{\,3}}{\cancel{4}_{\,2}}x^4 - \frac{4}{3}x^3 + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{24}\left(\frac{1}{6} - \frac{4}{5} + \frac{3}{2} - \frac{4}{3} + \frac{1}{2}\right)$$

$$= \frac{1}{24}\left(\frac{4}{2} - \frac{4}{5} + \frac{1}{6} - \frac{8}{6}\right)$$

$$= \frac{1}{24}\left(\frac{20}{10} - \frac{8}{10} - \frac{7}{6}\right)$$

$$= \frac{1}{24}\left(\frac{12}{10} - \frac{7}{6}\right)$$

$$= \frac{1}{24}\left(\frac{36}{30} - \frac{35}{30}\right)$$

$$= \frac{1}{24}\cdot\frac{1}{30}$$

$$= \frac{1}{720}$$

③ b) Bereken in cilindercoörd²:

$$\int_0^{a/\sqrt{2}} dy \int_y^{\sqrt{a^2-y^2}} \sqrt{x^2+y^2}\, dx \int_0^{\frac{x^2-y^2}{a}} dz \quad (a>0)$$

Cilindercoörd²: $x = r\cos\theta$ Jacobiaan: r

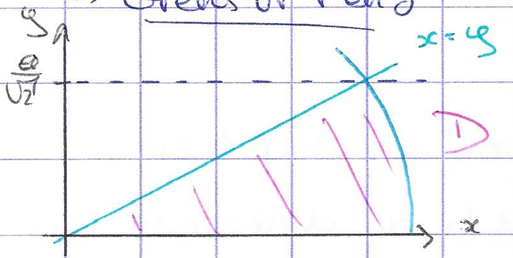
$y = r\sin\theta$

$z = z$

→ Grens vr z

$$0 \leq z \leq \frac{x^2-y^2}{a} \Rightarrow 0 \leq z \leq \frac{r^2\cos 2\theta}{a}$$

→ Grens vr r en θ



$$D = \{(x,y) \mid 0 \leq y \leq \frac{a}{\sqrt{2}},\ y \leq \underbrace{x \leq \sqrt{a^2-y^2}}_{*}\}$$

$* \Rightarrow x = \sqrt{a^2-y^2} \Leftrightarrow x^2 = a^2 - y^2 \Leftrightarrow x^2+y^2 = a^2$

(= vgl vd cirkel met straal a)

$$\Rightarrow D = \{(r\cos\theta, r\sin\theta) \mid 0 \leq \theta \leq \frac{\pi}{4},\ 0 \leq r \leq a\}$$

($\frac{\pi}{4}$ = arctan(rico v $x=y$) = arctan(1))

$$\Rightarrow I = \int_0^{\pi/4} d\theta \int_0^a r\, r\, dr \int_0^{\frac{r^2\cos 2\theta}{a}} dz$$

Jac

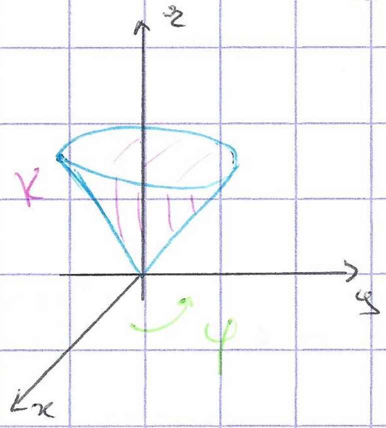
$$= \int_0^{\pi/4} d\theta \int_0^a \frac{r^4\cos 2\theta}{a}\, dr$$

$$= \int_0^{\pi/4} a^5 \frac{\cos 2\theta}{5a}\, d\theta$$

$$= \frac{a^4}{10}$$

4 b) Bereken en beklaard:

$$\iiint_K \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$$, met K begrensd dr $z=3$ en d kegel $x^2+y^2=z^2$.



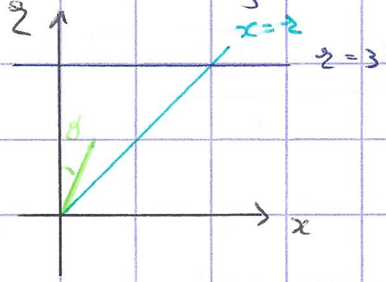
Transformatie:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

Grenzen:

→ vr φ omwentelingslich: $0 \leq \varphi \leq 2\pi$

→ dr figuur volstaat h om d dr snede met xz-vlak te bekijken



$\Rightarrow 0 \leq \theta \leq \frac{\pi}{4}$; $0 \leq r \leq \frac{3}{\cos\theta}$

$$\Rightarrow I = \int_0^{2\pi} d\varphi \int_0^{\pi/4} d\theta \, \underbrace{r^2\sin\theta}_{\text{Jac}} \int_0^{3/\cos\theta} r\,dr$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/4} \sin\theta\,d\theta \left[\frac{r^4}{4}\right]_0^{3/\cos\theta}$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/4} \sin\theta \, \frac{81}{4}\frac{1}{\cos^4\theta}\,d\theta$$

$u = \cos\theta \qquad du = -\sin\theta\,d\theta$

$$= -\int_0^{2\pi} d\varphi \int_0^{\pi/4} \frac{81}{4}\frac{1}{u^4}\,du$$

$$= -\int_0^{2\pi} d\varphi \, \frac{81}{4}\left[-\frac{1}{3}\frac{1}{u^3}\right]_{\cos 0}^{\cos\frac{\pi}{4}}$$

$$= +\int_0^{2\pi} d\varphi \, \frac{81}{12}\left(\left(\frac{2}{\sqrt{2}}\right)^3 - 1\right)$$

$$= 2\pi \cdot \frac{81}{12}\left(\frac{8}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} - 1\right)$$

$$= \frac{81}{6}\pi\left(\frac{4\sqrt{2}}{2} - 1\right)$$

$$= 27\pi\left(\sqrt{2} - \frac{1}{2}\right)$$

28/03 ⑤ d) Bereken h volume vh deel v $\{(x,y,z) \mid x^2+y^2 > a|x|\}$ ($a > 0$) dat begrensd wordt dr d sfeer $x^2+y^2+z^2=a^2$

Hoe ziet V eruit in xy-vlak

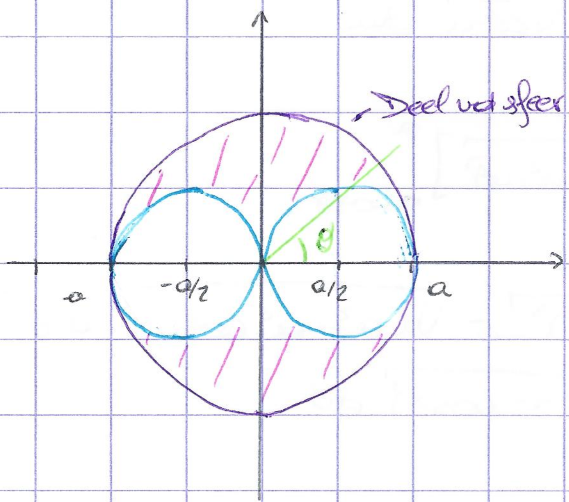
Geval 1 $x \geq 0$

$\Rightarrow x^2+y^2 > a|x| \Rightarrow x^2+y^2 > ax$ (vgl cirkel)

$\Leftrightarrow (x-\frac{a}{2})^2+y^2 > \frac{a^2}{4}$ (MP)

Geval 2 analoog of dr symm.

$\Rightarrow (x+\frac{a}{2})^2+y^2 > \frac{a^2}{4}$



Transform: cilindercoörd:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

We berekenen h vol vr $x > 0$ (Totale vol = 2·(vol vr $x>0$))

Grenzen vr r, θ en z:

→ $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

→ vr r: ondergrens komt v $x^2+y^2 = ax$

$\Rightarrow r^2 = ar\cos\theta \Leftrightarrow r = a\cos\theta$

bovengrens dr sfeer (in xy-vlak $z=0$)

$x^2+y^2=a^2 \Leftrightarrow r=a$

$\Rightarrow a\cos\theta \leq r \leq a$

→ vr z (vd sfeer)

$x^2+y^2+z^2 \leq a^2$

$\Rightarrow r^2+z^2 \leq a^2$

$\Rightarrow -\sqrt{a^2-r^2} \leq z \leq \sqrt{a^2-r^2}$

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$\Rightarrow Vol = \iiint r \, dx\,dy\,dz$

$$= \int_{-\pi/2}^{\pi/2} d\theta \int_{a\cos\theta}^{a} r\,dr \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dz$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \int_{a\cos\theta}^{a} r\,dr \,[z]_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}}$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \int_{a\cos\theta}^{a} r\left(\sqrt{a^2-r^2}+\sqrt{a^2-r^2}\right) dr$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \int_{a\cos\theta}^{a} 2r\sqrt{a^2-r^2}\,dr$$

$u = r^2 \qquad du = 2r\,dr$

$$= \int_{-\pi/2}^{\pi/2} d\theta \int_{a\cos\theta}^{a} \sqrt{a^2-u}\,du$$

$$= -\int_{-\pi/2}^{\pi/2} d\theta \; \frac{2}{3}\left[\sqrt{(a^2-r^2)^3}\right]_{a\cos\theta}^{a}$$

$$= -\frac{2}{3}\int_{-\pi/2}^{\pi/2} \sqrt{(a^2-a^2)^3} - \sqrt{(a^2-a^2\cos\theta)^3}\,d\theta$$

$$= +\frac{2}{3}\int_{-\pi/2}^{\pi/2} \sqrt{a^{2\cdot 3}(1-\cos\theta)^3}\,d\theta$$

$$= \frac{2a^3}{3}\int_{-\pi/2}^{\pi/2} (1-\cos\theta)^{3/2}\,d\theta$$

$= \ldots$

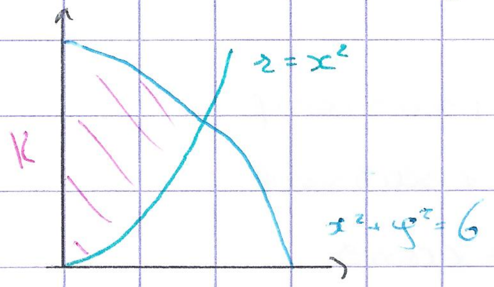
$$= \frac{8a^3}{9}$$

$\Rightarrow$ Tot^{e} volume $= 2\cdot\dfrac{8a^3}{9} = \dfrac{16a^3}{9}$

Extra oef

Bereken h vol vd bol $S \leftrightarrow x^2+y^2+z^2 \le 6$ binnen d paraboloïde $P \leftrightarrow z = x^2+y^2$

Dit is e omw-lich (x, y komen alleen vr als x^2+y^2)
Het volstaat om d figuur in xz-vlak te bekijken.
(De voll^e fig volgt dr rond z-as te wentelen).



Transformatie:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Grenzen vr z en r:

V is x-proj bi gebied

→ vr r: Bovengrens is h snijpt v $z = r$ en $z^2+x^2=6$

$\Rightarrow z + z^2 = 6$

$\Rightarrow z_{1,2} = \begin{cases} 2 \\ \cancel{-3} \end{cases} \Rightarrow r = \sqrt{2}$

$\Rightarrow 0 \le r \le \sqrt{2}$

→ vr z: $r^2 \le z \le \sqrt{6-r^2}$

$$\Rightarrow \text{Volume} = \iiint 1 \, dx\,dy\,dz$$
$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r \, dr \int_{r^2}^{\sqrt{6-r^2}} dz$$
$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r\left(\sqrt{6-r^2} - r^2\right) dr$$
$$= \int_0^{2\pi} d\theta \left(\int_0^{\sqrt{2}} \tfrac{2}{2} r\sqrt{6-r^2}\, dr - \int_0^{\sqrt{2}} r^3 \, dr \right) \quad u = r^2 \quad du = 2r\,dr$$
$$= \int_0^{2\pi} d\theta \left(\frac{1}{2} \int_0^{\sqrt{2}} \sqrt{6-u}\, du - \left[\frac{r^4}{4}\right]_0^{\sqrt{2}} \right)$$
$$= \int_0^{2\pi} d\theta \left(\frac{1}{2}\cdot\frac{2}{3} \left[(6-r^2)^{3/2}\right]_0^{\sqrt{2}} - \frac{2^{4/2}}{4} \right)$$
$$= -\int_0^{2\pi} d\theta \left(\frac{1}{3}\left((6-2)^{3/2} - 6^{3/2}\right) - 1 \right)$$
$$= -\frac{2\pi}{3} \left(\sqrt{4^3} - 6^{3/2} - 1 \right)$$
$$= -\frac{2\pi}{3} \left(7 - \sqrt{6^3} \right) = \frac{2\pi}{3}\left(6\sqrt{6} - 7 \right) \quad 11$$

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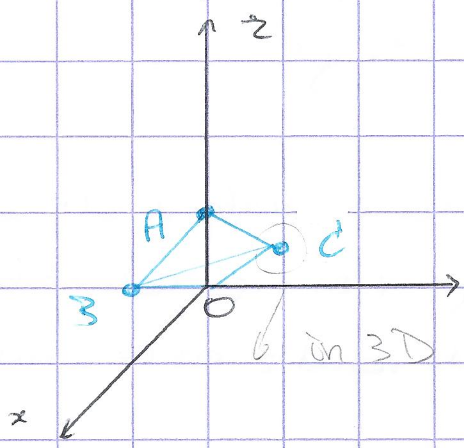
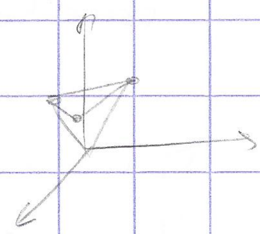
6 iii) Bereken vd piramide met hoekpt: $O(0,0,0)$, $A(0,0,1)$, $B(1,0,1)$ en $C(1,1,1)$ h vol an bolcoörd:





Transformatie:
$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r\cos\theta \end{cases}$$

Grens vr φ: φ loopt v L-vlak ($y=0$) tot R-vlak ($x=y$)

$$\Rightarrow \quad 0 \le \varphi \le \frac{\pi}{4}$$

Grens vr θ: θ loopt v 0 tot d grens uitgebracht dr h andervlak $z = x$ i.e. $r\cos\theta = r\sin\theta\cos\varphi$

$$\overset{r \neq 0}{\Leftrightarrow} \quad \tan\theta = \frac{1}{\cos\varphi} \Rightarrow \theta = \arctan\left(\frac{1}{\cos\varphi}\right)$$

$$\Rightarrow \quad 0 \le \theta \le \arctan\left(\frac{1}{\cos\varphi}\right)$$

Grens vr r: v 0 tot bovenvlak ($z=1$)

$$r\cos\theta = 1 \Leftrightarrow r = \frac{1}{\cos\theta}$$

$$\Rightarrow \quad 0 \le r \le \frac{1}{\cos\theta}$$

$$\Rightarrow \text{Volume} = \iiint 1 \, dx\,dy\,dz = \int_0^{\pi/4} d\varphi \int_0^{\arctan\left(\frac{1}{\cos\varphi}\right)} \sin\theta \, d\theta \int_0^{\frac{1}{\cos\theta}} r^2 \, dr$$

$$= \int_0^{\pi/4} d\varphi \int_0^{\arctan\left(\frac{1}{\cos\varphi}\right)} \frac{\sin\theta}{3\cos^3\theta} \, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{6} \, d\varphi \left[\frac{1}{\cos^2\theta}\right]_0^{\arctan\left(\frac{1}{\cos\varphi}\right)}$$

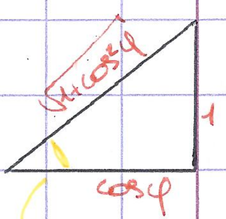
$$= \frac{1}{6} \int_0^{\pi/4} \left(\frac{1}{\cos^2\left(\arctan\left(\frac{1}{\cos\varphi}\right)\right)} - 1\right) d\varphi$$

$$= \frac{1}{6} \int_0^{\pi/4} \frac{1}{\frac{\cos^2\varphi}{1+\cos^2\varphi}} - 1 \, d\varphi$$

$$= \frac{1}{6} \int_0^{\pi/4} \frac{1+\cos^2\varphi}{\cos^2\varphi} - \frac{\cos^2\varphi}{\cos^2\varphi} \, d\varphi$$

$$= \frac{1}{6} \left[\tan\varphi\right]_0^{\pi/4}$$

$$= \frac{1}{6}$$



$\arctan\left(\frac{1}{\cos\varphi}\right)$

TOA

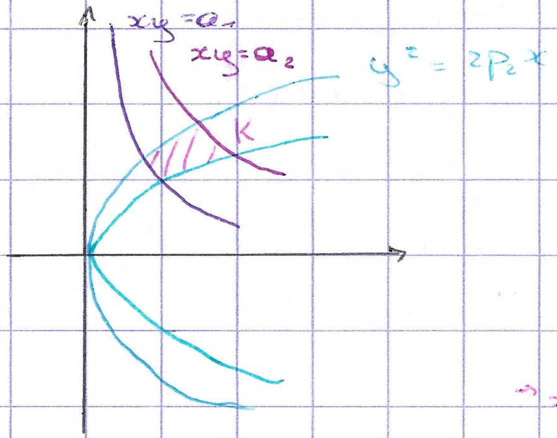
$\Rightarrow \tan\psi = \frac{O}{A}$

$\tan\psi = \frac{1}{\cos\varphi}$

$\psi = \arctan\left(\frac{1}{\cos\varphi}\right)$

Transformaties

① 4) Opp tussen: $y^2 = 2p_1 x$, $y^2 = 2p_2 x$, $xy = a_1$, $xy = a_2$



Transformatie

$$\Rightarrow \begin{cases} u = \dfrac{y^2}{2x} \\ v = xy \end{cases}$$

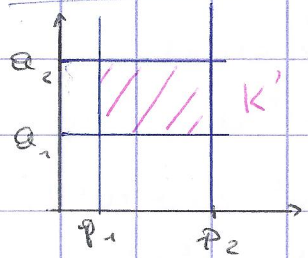
→ x elimineren: $u \cdot v = \dfrac{y^2}{2x} \cdot xy = \dfrac{y^3}{2}$

$$y = \sqrt[3]{2uv}$$

→ y elimineren: $\dfrac{v^2}{u} = \dfrac{x^2y^2}{y^2} \cdot 2x = 2x^3$

$$x = \sqrt[3]{v^2/2u}$$

Nieuw coörd. syst



$$\mathrm{Jac} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \left(-\frac{1}{3}\right)\frac{1}{\sqrt[3]{2}} v^{2/3} u^{-4/3} & \frac{2}{3}\frac{1}{\sqrt[3]{2}} v^{-\frac{1}{3}} u^{-\frac{1}{3}} \\ \frac{1}{3}\sqrt[3]{2}\, u^{-2/3} v^{\frac{1}{3}} & \frac{1}{3}\sqrt[3]{2}\, u^{\frac{1}{3}} v^{-2/3} \end{vmatrix}$$

$$= -\frac{1}{9}u^{-1} - \frac{2}{9}u^{-1}$$

$$= -\frac{1}{3u}$$

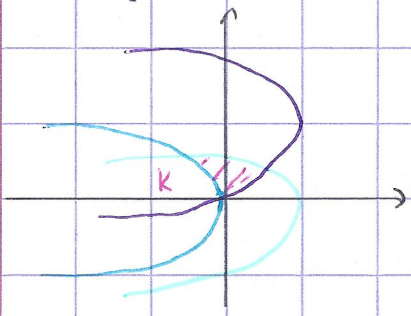
$$\iint_K 1\, dx\, dy = \iint_{K'} \left|-\frac{1}{3u}\right| du\, dv$$

$$= \int_{p_1}^{p_2} \frac{1}{3u} \int_{a_1}^{a_2} dv$$

$$= \frac{1}{3}(a_2 - a_1)\ln\left(\frac{p_2}{p_1}\right)$$

(36)

(7) $\iint_K x\,dx\,dy$ met K begrensd door $x = -y^2$, $x = 2y - y^2$ en $x = -2y - y^2 + 2$



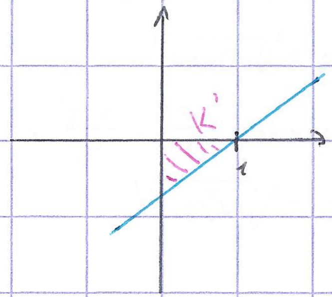
Transformatie

$$\begin{cases} u = x + y^2 & (x = -y^2 \text{ met } u = 0) \\ v = x + y^2 - 2y & (x = 2y - y^2 \text{ met } v = 0) \end{cases}$$

$$\Rightarrow \begin{cases} x = u - \left(\frac{u-v}{2}\right)^2 \\ y = \frac{u-v}{2} \end{cases}$$

Nieuwe vgl

$u = 0,\ v = 0,\ 2u - v = 2$



$$\text{Jac} = \begin{vmatrix} 1 - \left(\frac{u-v}{2}\right) & \frac{u-v}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\Rightarrow \iint_K x\,dx\,dy = \iint_{K'} \left(u - \left(\frac{u-v}{2}\right)^2\right) \left|-\frac{1}{2}\right| du\,dv$$

$$= \frac{1}{2} \int dv \int u - \left(\frac{u-v}{2}\right)^2 du$$

$$= \frac{1}{2} \int dv \cdot \left(\left[\frac{u^2}{2}\right] - \int\left(\frac{1}{4} \cdot (u^2 - 2uv + v^2)\right) du\right.$$

$$= \frac{1}{2} \int dv \cdot \left(\frac{1}{2} - \frac{1}{4}\left(\left[\frac{u^3}{3} - 2v\frac{u^2}{2}\right]_0^1\right.\right.$$

$$= \frac{1}{2} \int \frac{1}{2} dv - \frac{1}{8} \int \left(\frac{1}{3} - v\right) dv$$

$$= \frac{1}{4} - \frac{1}{8} \cdot \frac{1}{3} - \frac{1}{8} \cdot \frac{1}{2}$$

$$= \frac{1}{4} - \frac{1}{24} - \frac{1}{16}$$

$$= \frac{6}{24} - \frac{1}{24} - \frac{1}{16}$$

$$= \frac{5}{24} - \frac{1}{16}$$

$$= \frac{10}{48} - \frac{3}{48}$$

?

$$= \frac{1}{48}$$

6 Volume ingesloten dr $z = x^2 + 2y^2$ en $z = y$

Transformatie

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \cdot \frac{1}{\sqrt{2}} \\ z = z \end{cases}$$

→ x en z : cilindercoörd² en deze gebruiken om y te vinden met $z = x^2 + y^2$

$\Rightarrow$ Vgl² $\Leftrightarrow$: $z = r^2$ en $z = \frac{r\sin\theta}{\sqrt{2}}$

Volume wordt bepaald dr $z \geq x^2 + 2y^2$ en $z \leq y$

$\Rightarrow z \geq r^2$ en $z \leq \frac{r\sin\theta}{\sqrt{2}}$ (= Grenzen vr z)

Grenzen

$\Rightarrow r^2 \leq z \leq \frac{r\sin\theta}{\sqrt{2}}$

$\Rightarrow$ ondergrens steeds kleiner dan bovengrens

Extra vw: $r^2 \leq \frac{r\sin\theta}{\sqrt{2}}$

$\Leftrightarrow r \leq \frac{\sin\theta}{\sqrt{2}}$ bovengrens vr r is in functie

→ Grens vr r: $0 \leq r \leq \frac{\sin\theta}{\sqrt{2}}$

Extra vw: $0 \leq \frac{\sin\theta}{\sqrt{2}} \Leftrightarrow \sin\theta$ pos

$\Rightarrow 0 \leq \theta \leq \pi$

$$Jac = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \frac{\sin\theta}{\sqrt{2}} & \frac{r\cos\theta}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{r\cos^2\theta}{\sqrt{2}} + \frac{r\sin^2\theta}{\sqrt{2}} = \frac{r}{\sqrt{2}}$$

$$\Rightarrow \text{Volume} = \iiint_V 1\,dx\,dy\,dz = \int_0^\pi d\theta \int_0^{\frac{\sin\theta}{\sqrt{2}}} \frac{r}{\sqrt{2}}\,dr \int_{r^2}^{\frac{r\sin\theta}{\sqrt{2}}} dz$$

$$= \int_0^\pi d\theta \int_0^{\frac{\sin\theta}{\sqrt{2}}} \frac{r}{\sqrt{2}}\left(\frac{r\sin\theta}{\sqrt{2}} - r^2\right) dr$$

$$= \int_0^\pi d\theta \int_0^{\sin\theta/\sqrt{2}} \frac{r^2\sin\theta}{2} - \frac{r^3}{\sqrt{2}}\,dr = \int_0^\pi d\theta \left[\frac{r^3\sin\theta}{6} - \frac{r^4}{4\sqrt{2}}\right]_0^{\frac{\sin\theta}{\sqrt{2}}}$$

$$= \int_0^\pi \frac{\sin^3\theta \sin\theta}{6(\sqrt{2})^3} - \frac{\sin^4\theta}{4(\sqrt{2})^4}\,d\theta = \int_0^\pi \sin^4\theta\left(\frac{1}{12\sqrt{2}} - \frac{1}{256}\right) d\theta$$

$= \ldots = \frac{\sqrt{2}\,\pi}{256}$ → checken op minerva

38 2004 H' dst Fuchs dis

$P(x,y) + Q(x,y)\,y' = 0$ (1)

waar P en Q glad op G, met G open gebied znd gaten

"Formeel"

$P(x,y)\,dx + Q(x,y)\,dy = 0$

Beschouw vectorveld $\vec{F} = (P, Q)$

Geval 1

$\vec{F}$ is e wervelvrij vectorveld $\overset{\text{zie deel } a_j}{\Longleftrightarrow}$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\overset{\text{deel } a_j}{\Longrightarrow}$ $\vec{F}$ is gradiënt v scalaren veld f

$(\exists f : G \subseteq \mathbb{R}^2 \to \mathbb{R})\left(\frac{\partial f}{\partial x} = P, \frac{\partial f}{\partial y} = Q\right)$

→ Diff vgl kan geschreven w als $\frac{\partial}{\partial x}(f(x,y)) = 0$

⇒ Opl v (1) w dan geg dr $f(x,y) = c$

met c e will^e cte

vb

$y\,dx + 2x\,dy = 0$

(Stelt $y + 2xy' = 0$ voor)

$P(x,y) = y$ $Q(x,y) = 2x$

→ Nagaan of (P, Q) wervelvrij $1 = \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} = 2$

⇒ (P, Q) is nt wervelvrij

→ Vermenigvuldig diff-vgl met y

⇒ $y^2\,dx + 2xy\,dy = 0$

⇒ $\tilde{P} = y^2$ $\tilde{Q} = 2xy$

→ Nagaan of $(\tilde{P}, \tilde{Q})$ wervelvrij

$\frac{\partial \tilde{P}}{\partial y} = 2y = \frac{\partial \tilde{Q}}{\partial x} = 2y$

⇒ $(\tilde{P}, \tilde{Q})$ is wervelvrij, zoek f

→ Uit $\frac{df}{dx} = \tilde{P} = y^2$ ⇒ $f(x,y) = xy^2 + A(y)$. $\forall x,y$

met $A(y)$ e cte die mag afhangen v y.

Uit $\frac{df}{dy} = \tilde{Q} = 2xy$ ⇒ $f(x,y) = xy^2 + B(x)$. $\forall x,y$

⇒ Zelfde xy^2 bij $\tilde{P}$ en $\tilde{Q}$ ⇒ dit drdat wervelvrij.

→ Kan $y = y_0$

$f(x,y_0) = xy_0^2 + A(y_0)$

$= xy_0^2 + B(x)$ $\forall x$

⇒ $A(y_0) = B(x)$ $\forall x$

maw $B(x)$ is ct (nt afh v x)

→ Analoog vr $A(y)$ is ct (nt afh v y)

⇒ $f(x,y) = xy^2 + c$

Conclusie: Opl^2 w geg. dr $f(x,y) = c$ met c e willk cte

⇒ $xy^2 = c$ " "

(⇔ $y = \frac{c}{\sqrt{x}}$ is een opl.)

Geval 2

(P,Q) is nt wervelvrij.

Vermenigvuldig met fct μ zodat $(\mu P, \mu Q)$ wel wervelvrij

⇒ Ga verder zoals in geval 1)

40 Oef p3

1(d), 2, 3(c,d)

① d) $(3x^2y + 8xy^2)\,dx + (x^3 + 8x^2y + 12ye^y)\,dy = 0$

$P(x,y) = 3x^2y + 8xy^2$

$\frac{\partial P}{\partial y} = 3x^2 + 16xy$

$Q(x,y) = x^3 + 8x^2y + 12ye^y$

$\frac{\partial Q}{\partial x} = 3x^2 + 16yx$

$\Rightarrow (P, Q)$ is wervelvrij

Zoek f

$\rightarrow \frac{df}{dx} = 3x^2y + 8xy^2 \Rightarrow f(x,y) = yx^3 + 4x^2y^2 + A(y)$

$\rightarrow \frac{df}{dy} = x^3 + 8x^2y + 12ye^y \Rightarrow f(x,y) = x^3y + 4x^2y^2 + 12(y-1)e^y + B(x)$

$(\int ye^y\,dy = \int y\,de^y = e^y y - \int e^y\,dy = e^y y - e^y = (y-1)e^y)$

$\Rightarrow \forall x,y \quad A(y) = 12(y-1)e^y + B(x)$

$\Longrightarrow A(y) = 12(y-1)e^y + C_1$

$B(x) = C_1$

$\Rightarrow$ Opl^e is geg dr

$x^3y + 4x^2y^2 + 12(y-1)e^y = c$ met c e will^e cte

wss examen !

② Toon aan dat μ is integrerende factor als

(*) $\frac{1}{\mu}\left(Q\frac{\partial\mu}{\partial x} - P\frac{\partial\mu}{\partial y}\right) = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$

a) Als $f(x) := \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ nt afh v y $\Rightarrow e^{\int f(x)\,dx}$ is integr^e factor

b) Als $g(y) := \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ nt afh v x $\Rightarrow e^{\int g(y)\,dy}$

μ is integr^e factor als $(\mu P, \mu Q)$ wervelvrij

$\Leftrightarrow \frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}$

Kettingregel

$\Leftrightarrow P\frac{\partial\mu}{\partial y} + \mu\frac{\partial P}{\partial y} = Q\frac{\partial\mu}{\partial x} + \mu\frac{\partial Q}{\partial x}$

$\Leftrightarrow \frac{1}{\mu}\left(Q\frac{\partial\mu}{\partial x} - P\frac{\partial\mu}{\partial y}\right) = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$

(41)

a) $\mu = e^{\int f(x)\,dx}$ integr. factor

$\frac{\partial \mu}{\partial x} = e^{\int f(x)dx} f(x)$ $\frac{\partial \mu}{\partial y} = 0$

(1) $\Leftrightarrow \frac{1}{\mu}(Q\mu f(x)) = \underbrace{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}$

$\Rightarrow Q f = Qf$

$\Rightarrow$ (1) is voldaan

$\Rightarrow$ μ is integr° factor

b) Analoog

3 c) $y\,dx + (x^2y - x)\,dy = 0$

$P(x,y) = y$ $Q(x,y) = x^2y - x$

$\frac{\partial P}{\partial y} = 1 \neq \frac{\partial Q}{\partial x} = 2yx - 1$ ((P,Q) geen wervelvrij veld)

opl (2)

$\Rightarrow$ Zoek integr° factor

$f(x) = \frac{1}{x^2y - x}(1 - 2yx + 1)$

$= \frac{1}{x(xy-1)}(2 - 2xy)$

$= \frac{-2(\cancel{yx-1})}{x(\cancel{yx-1})}$

$= \frac{-2}{x}$

$\Rightarrow e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx}$ is integr° factor

$\Rightarrow \int f(x)dx = \int -\frac{2}{x}dx = -2\int \frac{1}{x}dx = -2\ln x$

$\Rightarrow e^{-2\ln x} = (e^{\ln x})^{-2} = x^{-2} = \frac{1}{x^2}$

$\Rightarrow$ Vmv diffvgl met integr° factor $\frac{1}{x^2}$

$\tilde{P}(x,y) = \frac{y}{x^2}$ $\tilde{Q}(x,y) = y - \frac{1}{x}$

$\frac{\partial \tilde{P}}{\partial y} = \frac{1}{x^2} = \frac{\partial \tilde{Q}}{\partial x} = \frac{1}{x^2}$

$\Rightarrow$ exact

$\Rightarrow$ Zoek φ zodat

$$\frac{\partial \varphi}{\partial x} = \tilde{P} = \frac{y}{x^2} \quad \Rightarrow \quad \varphi(x,y) = -\frac{y}{x} + A(y).$$

$$\frac{\partial \varphi}{\partial y} = \tilde{Q} = y - \frac{1}{x} \quad \Rightarrow \quad \varphi(x,y) = -\frac{y}{x} + \frac{y^2}{2} + B(x).$$

$$\Rightarrow \forall x,y \qquad A(y) = \frac{y^2}{2} + B(x)$$

$$\Rightarrow \quad A(y) = \frac{y^2}{2} + C_1$$

$$B(x) = C_1$$

$\Rightarrow$ Opl$^{\text{e}}$ v. gegeven d.v.

$$-\frac{y}{x} + \frac{y^2}{2} = c \qquad \text{met } c \text{ willek}^{\text{e}} \text{ cte}$$

$$\Leftrightarrow \frac{y^2}{2} - \frac{y}{x} = c$$

e) $y' + a(x)y = R(x)$ met a en R glad

$\cancel{\Leftrightarrow dy + a(x)y\,dx - R(x) = 0}$

$\cancel{P(x,y) = +a(x)y - R(x) \qquad Q(x,y) = 1}$

$\cancel{\frac{\partial P}{\partial y} = a(x) \qquad \frac{\partial Q}{\partial x} = 0}$

Formeel : $(a(x)y - R(x))\,dx + dy = 0$

$$P(x,y) = a(x)y - R(x) \qquad Q(x,y) = 1$$

$$\frac{\partial P}{\partial y} = a(x) \neq \frac{\partial Q}{\partial x} = 0$$

Zoek integr$^{\text{e}}$ factor

$$f(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = a(x) \quad \text{hangt niet af v. } y$$

$$\Rightarrow e^{\int a(x)dx}$$

Vmv diff.vgl. met integr. factor

$$e^{\int a(x)dx}(a(x)y - R(x))\,dx + e^{\int a(x)dx}\,dy = 0$$

$$\left.\begin{aligned}\tilde{P}(x,y) &= e^{\int a(x)dx}(a(x)y - R(x)) \\ \tilde{Q}(x,y) &= e^{\int a(x)dx}\end{aligned}\right\} \quad (\tilde{P},\tilde{Q}) \text{ is wervelvrij}$$

Zoek f zodat $\vec{\nabla} f = (\tilde{P}, \tilde{Q})$

$$\frac{\partial f}{\partial x} = e^{\int a(x)dx}(a(x)y - R(x)) \quad \Rightarrow f(x,y) = e^{\int a(x)dx}y - \int R(x)e^{\int a(x)dx}dx + A(y)$$

$$\frac{\partial f}{\partial y} = e^{\int a(x)dx} \quad \Rightarrow f(x,y) = e^{\int a(x)dx}y + B(x)$$

$\Rightarrow f(x,y) = e^{\int a(x)dx} y - \int R(x) e^{\int a(x)dx} dx + C$

$\Rightarrow$ Opl^t vd diff.vgl is geg. dr

$$e^{\int a(x)dx} y - \int R(x) e^{\int a(x)dx} dx = c \quad \text{met } c \text{ willek}^e \text{ cte}$$

$$\Rightarrow y(x) = e^{-\int a(x)dx} \left(c + \int R(x) e^{\int a(x)dx} dx \right) \text{ met } c \text{ " "}$$

Theorie nieuw deel

We zoeken nt-triviale opl. v

$$\boxed{y'' + \lambda y = 0}$$

met randvw^e $\quad y(0) = 0 \quad , \quad y(L) = 0 \quad (L \neq 0)$

Vr welke λ?

Geval 1 $\quad \lambda < 0$

Alg^e opl. is $\quad A e^{\sqrt{-\lambda}\, x} + B e^{-\sqrt{-\lambda}\, x}$

Randvw^e geven $\begin{cases} A + B = 0 \\ A e^{\sqrt{-\lambda}\, L} + B e^{-\sqrt{-\lambda}\, L} = 0 \end{cases}$

Er is e nt-triviale opl. als

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\, L} & e^{-\sqrt{-\lambda}\, L} \end{vmatrix} = 0 \Leftrightarrow e^{-\sqrt{-\lambda}\, L}\underbrace{\left(1 - e^{2\sqrt{-\lambda}\, L}\right) = 0}_{\text{gebeurt nooit}}$$

$\Rightarrow$ enkel triv^e opl

Geval 2 $\quad \lambda = 0$

Alg^e opl is $\quad Ax + B$

Randvw^e geven $\begin{cases} B = 0 \\ AL + B = 0 \end{cases} \Rightarrow A = B = 0 \Rightarrow$ enkel triv^e opl

Geval 3 $\quad \lambda > 0$

Alg^e opl. is $\quad A \sin(\sqrt{\lambda}\, x) + B \cos(\sqrt{\lambda}\, x)$

Randvw^e geven $\begin{cases} y(0) = 0 \Rightarrow B = 0 \\ y(L) = 0 \Rightarrow A \sin(\sqrt{\lambda}\, L) = 0 \end{cases}$

We zoeken e nt-triviale opl. $\quad \Rightarrow A \neq 0$

$\Rightarrow \sin(\sqrt{\lambda}\, L) = 0 \quad \Rightarrow \sqrt{\lambda}\, L = n\pi \quad$ vr $n \in \mathbb{N}_+$

$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \quad$ vr $n \in \mathbb{N}_+$

Conclusie

Het randvw^e-probl. h e nt-triv^e opl $\Leftrightarrow$

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{vr e } n \in \mathbb{N}_+$$

De λ_n zijn d eigenwaarden.
Vr deze λ_n heb je d niet-triv^e opl.

$$\varphi_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right)$$ vr will^e A_n

Def p5

$\varphi'' + \lambda\varphi = 0 \qquad \varphi'(0) = \varphi'(L) = 0 \qquad (L > 0)$

Geval 1 $\lambda < 0$

Alg^e opl. $\quad \varphi(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

$\Rightarrow \varphi'(x) = A\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$

$0 = \varphi'(0) = \sqrt{-\lambda}(A - B)$

$\Rightarrow A = B$

$0 = \varphi'(L) = \sqrt{-\lambda}\left(Ae^{\sqrt{-\lambda}L} - Be^{-\sqrt{-\lambda}L}\right)$

$\Rightarrow A\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right) = 0$

$\Rightarrow A = 0 \qquad (\lambda \neq 0 \neq L)$

$\Rightarrow B = 0$

$\Rightarrow$ enkel triv^e opl.

Geval 2 $\lambda = 0$

Alg^e opl. $\quad \varphi(x) = Ax + B$

$\Rightarrow \varphi'(x) = A$

$\overset{\text{RVW}}{\Rightarrow} A = 0$

$\Rightarrow \lambda = 0$ is eigenwaarde met eigenfcts
$\varphi(x) = B$ met B will. constant

Geval 3 $\lambda > 0$

Alg^e opl. $\quad \varphi(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$

$\Rightarrow \varphi'(x) = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$

Randvw^e $\quad 0 = \varphi'(0) = A\sqrt{\lambda} \Rightarrow A = 0$

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$$0 = \varphi'(L) = -B\sqrt{\lambda}\sin(\sqrt{\lambda}L)$$

We zoeken e nt-triv^e opl $\Rightarrow B \neq 0$

$\Rightarrow \quad \sin(\sqrt{\lambda}L) = 0$

$\Rightarrow \sqrt{\lambda}L = n\pi \qquad$ vr $n \in \mathbb{N}_+$

$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \qquad$ vr $n \in \mathbb{N}_+$

$\Rightarrow$ vr $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$ zijn er d nt-triv^e opl.

$$\varphi_n(x) = B_n \cos\left(\frac{n\pi x}{L}\right)$$

Toep.

We zoeken opl. vd part diff vgl

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad (x,t) \in [0,L] \times [0,\infty)$$

met randVW:

$$u(0,t) = u(L,t) = 0 \qquad \forall t \in [0,\infty)$$

en beginVW

$$u(x,0) = f(x)$$

beginsnelheid

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

We bepalen opl vd vorm $\quad u(x,t) = \varphi(x) z(t) \quad \forall x,t$

$\overset{PD}{\Rightarrow} \quad \alpha^2 \varphi''(x) z(t) = \varphi(x) z''(t)$

$\Rightarrow \quad \dfrac{\varphi''(x)}{\varphi(x)} = \dfrac{z''(t)}{\alpha^2 z(t)} =: -\lambda$

$$\Rightarrow \begin{cases} \varphi'' + \lambda\varphi = 0 \\ z'' + \alpha^2\lambda z = 0 \end{cases}$$

Uit d randvw $\Rightarrow \quad \varphi(0)z(t) = 0 = \varphi(L)z(t) \quad \forall t$

$\Rightarrow \quad z(t) = 0 \quad \forall t$

of $\varphi(0) = 0 = \varphi(L)$

$\rightarrow u(x,t) = 0 \quad \forall x,t$

$\Rightarrow$ Vorige paragraaf

$\Rightarrow \quad \varphi(x) = A_n \sin\left(\frac{n\pi x}{L}\right)$ en $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n \in \mathbb{N}$

opl $\overset{\text{VI}^2}{\Rightarrow} \quad z_n(t) = \tilde{A}_n \cos\left(\frac{\alpha n\pi t}{L}\right) + \tilde{B}_n \sin\left(\frac{\alpha n\pi t}{L}\right) \quad n \in \mathbb{N}_+$

$$\Rightarrow u_n(x,t) = \bar{A}_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{\alpha n\pi t}{L}\right) + \bar{B}_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{\alpha n\pi t}{L}\right)$$

vr willk. $\bar{A}_n, \bar{B}_n$

$\Rightarrow$ Vervolg

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad (x,t) \in [0,L] \times [0,+\infty)$$

$$\text{RVW:} \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

$$\text{Begin VW:} \begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

Samenvatting vorige les

opl. vd vorm $u(x,t) = \varphi(x)\, z(t)$

$$u_n(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi\alpha t}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\alpha t}{L}\right) \quad n \in \mathbb{N}$$

$\Rightarrow$ We maken d alg. opl:

$$u(x,t) = \sum_{n=1}^{+\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi\alpha t}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\alpha t}{L}\right)$$

RandVW voldaan?

$$u(x,0) = \sum_{n=1}^{+\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi\alpha}{L} = g(x)$$

Analyse I

$$\Rightarrow \begin{cases} a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ b_n = \frac{2}{n\pi\alpha}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases}$$

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p7

1 b) $g(x) = 0$
$L = \pi$
$f(x) = \frac{1}{\pi} x(\pi - x)$

RVW $u(0,t) = 0$
$u(\pi, t) = 0$
BVW $u(x,0) = 0$
$\frac{\partial u}{\partial x}(x,0) = \frac{1}{\pi} x(\pi - x)$

$\Rightarrow$ even/oneven?

$f(-x) = \frac{1}{\pi}(-x)(\pi + x) \neq -f(x)$

$\rightarrow g(x) = 0 \Rightarrow b_n = 0$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi \frac{1}{\pi} x(\pi - x) \sin(nx)\,dx$

$= \frac{2}{\pi^2}\int_0^\pi (\pi x - x^2)\sin(nx)\,dx$

$= \frac{2}{\pi^2}\left(\int_0^\pi \pi x \sin(nx)\,dx - \int_0^\pi x^2 \sin(nx)\,dx\right)$

(2) (1)

(1) $du = \sin(nx)\,dx \rightarrow u = -\frac{1}{n}\cos(nx)$

$v = x^2 \rightarrow dv = 2x\,dx$

$\Rightarrow \left[-\frac{x^2}{n}\cos(nx)\right]_0^\pi + \int_0^\pi \frac{2x}{n}\cos(nx)\,dx$

$u = \frac{1}{n}\sin(nx) \qquad dv = \frac{2}{n}dx$

$= -\frac{x^2}{n}\cos(nx) + \left[\frac{2x}{n^2}\sin(nx)\right]_0^\pi - \int_0^\pi \frac{2}{n^2}\sin(nx)\,dx$

Korte opl: volgende pag.

$$\int_{bol} dS = 2\pi r^2 \int_0^{\pi} \sin^3 x \cos x$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \frac{1}{\pi} x(\pi - x) \sin(nx)\, dx$$

$$= \frac{2}{\pi^2} \left[\frac{-x(\pi - x)\cos(nx)}{n} \right]_0^{\pi} + \frac{2}{\pi^2} \int_0^{\pi} \frac{\cos(nx)}{n} (-2x + \pi)\, dx$$

$$= -\frac{4}{\pi^2 n} \left[\frac{x \sin(nx)}{n} \right]_0^{\pi} + \frac{4}{\pi^2 n} \int_0^{\pi} \frac{\sin(nx)}{n}\, dx$$

$$= \frac{4}{\pi^2 n^2} \left[-\frac{\cos(nx)}{n} \right]_0^{\pi} \qquad -(-1) + 1 = 0$$

$\rightarrow$ n even: $\quad 0 \qquad (\cos 2\pi + \cos 0) \ldots$

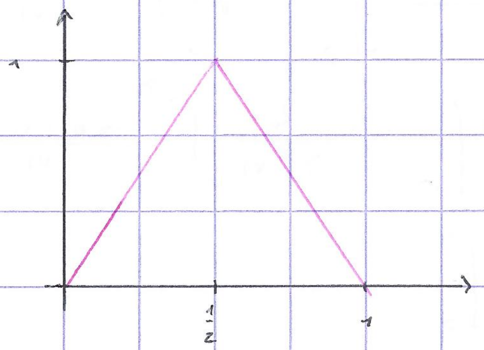
$\rightarrow$ n oneven: $\quad \frac{8}{\pi^2 n^3}$

$\Rightarrow$ opl volgt dr a_n en b_n in te vullen in

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi \alpha t}{L}\right) + \underbrace{b_n}_{=0} \ldots$$

2 $L=1$ $\quad f(x)=0$ $\quad g(x)=\begin{cases} 2x & x\in[0,\frac{1}{2}] \\ 2(1-x) & x\in[\frac{1}{2},1] \end{cases}$

$\alpha=1$



$a_n=0 \quad \forall n \quad (f(x)=0)$

$$b_n=\frac{2}{n\pi}\int_0^{1/2} 2x\sin\left(\frac{n\pi x}{L}\right)dx+\frac{2}{n\pi}\int_{1/2}^{1} 2(1-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{2}{n\pi}\int_0^{1/2} 2x\sin(n\pi x)\,dx+\frac{2}{n\pi}\int_{1/2}^{1} 2t\sin(n\pi(1-t))\,dt$$

$\to$ wisselend

$$=\frac{2}{n\pi}\int_0^{1/2} 2x\sin(n\pi x)\,dx+\frac{2}{n\pi}\int_{1/2}^{1}(-1)^{n+1}\,2t\sin(n\pi t)\,dt$$

$\to$ n even: $b_n=0$

$\to$ n oneven: $b_n=\frac{8}{n\pi}\int_0 t\sin(n\pi t)\,dt$

$$=\frac{8}{n\pi}\left[\frac{t(-\cos(n\pi t))}{n\pi}\right]_0^{1/2}+\frac{8}{n\pi}\int_0^{1/2}\frac{\cos(n\pi t)}{n\pi}\,dt$$

$$=\frac{8}{n^2\pi^2}\left[\frac{\sin(n\pi t)}{n\pi}\right]_0^{1/2}$$

$$=\frac{8}{n^3\pi^3}\,(-1)^{\frac{n-1}{2}}$$