(Substitutie 24-10-16

5 | 6, 11, 13, 15, 17

Voorbeeldoefening:

5 | 2

$$\int \frac{e^x}{1+e^{2x}}dx = \int \frac{1}{1+u^2}\,du$$

~~du = arctan(u)~~
~~= arctan(e^x)~~
~~u = e^x~~

$$\boxed{\begin{aligned} u &= e^x \\ du &= (e^x)'dx = e^x dx \end{aligned}}$$

$= \arctan(u) = \arctan(e^x)$

6 $\int \frac{\cos x}{\sin^2 x}dx = \int \frac{1}{u^2}du = -u^{-1}$
$= \frac{-1}{\sin x} + C$

$$\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x\, dx \end{aligned}}$$

11 $\int \frac{e^{-2/x^2}}{x^3}dx = \int \frac{e^u}{x^3}du \cdot x^3/4$
$= \frac{1}{4}\int e^u du$
$= \frac{1}{4}e^u + C$
$= \frac{1}{4}e^{-2/x^2} + C$

$$\boxed{\begin{aligned} u &= -2/x^2 \\ du &= 4/x^3\,dx \\ dx &= du \cdot x^3/4 \end{aligned}}$$

13 $\int \frac{x}{\sqrt{1-x^2}}dx$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned}} \quad \boxed{\begin{aligned} v &= 1-u \\ dv &= -du \end{aligned}}$$

$= \frac{1}{2}\int \frac{1}{\sqrt{1-u}}du$
$= -\frac{1}{2}\int \frac{1}{\sqrt{v}}dv$
$= -\frac{1}{2}\frac{\sqrt{v}}{1/2} = -\sqrt{v} = \sqrt{1-x^2} + C$

[

$\boxed{15}$ $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx$

$$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}}$$

$\sin^2 x = 1 - \cos^2 x = 1 - u^2$

$$= -\int (1-u^2)\,du$$
$$= -u + \frac{u^3}{3} = -\cos x + \frac{\cos^3 x}{3}$$

$\boxed{17}$ $\int \frac{x}{(a^2+x^2)\ln(a^2+x^2)}$

$$\boxed{\begin{array}{l} u = \ln(a^2+x^2) \\ du = \frac{2x}{a^2+x^2}\,dx \end{array}}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln|\ln|(a^2+x^2)|| + C$$

~~Oefening~~ ~~$\boxed{7}$~~ $\boxed{7}$

~~$\boxed{7}$ $\boxed{1}$ 8 9~~

$\boxed{7}$ $\int \frac{e^{-2x}}{1-e^{-x}}\,dx$

$$\boxed{\begin{array}{l} \text{Stel: } e^{-x} = t \\ \quad -e^{-x}dx = dt \end{array}}$$

$$= \int \frac{-t+1-1}{1+t}\,dt = -\int \frac{t+1}{t+1}\,dt + \int \frac{dt}{1+t}$$
$$= -t + \ln|1+t|$$
$$= \cancel{ee} - e^{-x} + \ln(1+e^{-x}) + C$$

~~Partiële integratie~~

25-10-16

$\boxed{6}$ $\boxed{1}$ $\int x\sin x\,dx$

$$\boxed{\begin{array}{ll} u = x & du = 1 \\ dv = \sin x\,dx & v = -\cos x \end{array}}$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

$\boxed{2}$ $\int \arctan x\,dx$

$$= \int 1 \cdot \arctan x\,dx$$

$$\boxed{\begin{array}{ll} \text{stel} & 1 + x^2 = t \\ & 2dx = dt \end{array}}$$

$$= x\arctan x - \int \frac{1}{1+x^2}\,dx$$

$$= x\arctan x - \int \frac{1}{2}\frac{dt}{t}$$

$$= x\arctan x - \frac{\ln(1+x^2)}{2} + C$$

$\boxed{4}$ $\boxed{a \neq -1}$

$$\int x^a \ln a\,dx = \frac{x^{a+1}\ln a}{a+1} - \frac{1}{a+1}\int x^a\,dx$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = \frac{1}{x} \\ dv = x^2dx & v = \frac{x^{a+1}}{a+1} \end{array}}$$

$$= \frac{x^{a+1}\ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2}$$

$\boxed{a = -1}$

$$\int \frac{\ln x}{x}\,dx$$

$$= \int \ln x\,d\ln x$$

$$= \frac{\ln^2 x}{2}$$

→ blijven doen tot dat de veelterm volledig weg is.

$\boxed{5}$ $\int x^3 e^{2x} dx$

$$\boxed{\begin{array}{ll} u = x^3 & du = 3x^2 dx \\ dv = e^{2x} dx & v = \frac{e^{2x}}{2} \end{array}}$$

$$= \int x^2 e^{2x} dx \quad \boxed{\begin{array}{ll} u = x^2 & du = 2x dx \\ dv = e^{2x} dx & v = \frac{e^{2x}}{2} \end{array}}$$

$$= \frac{x^2 e^{2x}}{2} - \left(\frac{1}{2} x e^{2x} - \frac{1}{2} \int e^x dx\right)$$

$$= \frac{x^2 e^{2x}}{2} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2x} + C$$

$\boxed{6}$ $\int \sin 2x \, e^{\sin x} dx$

$$= \int 2 \sin e^{\sin x} \cos x dx$$

$$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x dx \end{array}}$$

$$= 2 \int u \cdot e^u du$$

$$= 2 \cdot (u \cdot e^u - \int e^u)$$

$$= 2e^u (u - 1)$$

$$= 2ue^u - 2e^u$$

$$= 2e^{\sin x} \sin x - 2e^{\sin x}$$

$\boxed{8}$ $\int e^x \sin x dx$

$$= -\cos x \, e^x + \int e^x \cos x \, dx$$

$$\boxed{\begin{array}{ll} e^x = u & e^x dx = du \\ \sin x dx = dv & v = \cos x \end{array}}$$

$$= -\cos x e^x + \sin x e^x \underline{- \int e^x \sin x dx}$$

→ weg en * 2 aan de linkerkant, delen door 2.

$$= \frac{e^x (\sin x - \cos x)}{2}$$

altijd een graad verlagen

$\boxed{7}$ $\boxed{1}$ $I_n = \int \sin^n x \, dx \qquad n \geqslant 2$

$= \int \sin^{n-1} x \sin x \, dx$

$$\boxed{\begin{array}{ll} u = \sin^{n-1} x & du = (n-1)\sin^{n-2} x \cos x \\ dv = \sin x \, dx & v = -\cos x \end{array}}$$

$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \underbrace{\cos^2 x}_{= 1 - \sin^2 x} dx$

$= -\cos x \sin^{n-1} x + (n-1) \int \sin^n x \, dx$

$\Rightarrow I_n = \frac{1}{n}\left(-\cos x \sin^{n-1} x + (n-1) I_{n-2}\right)$

$I_0 = x \quad I_1 = -\cos x$

26-10-16

Rationeel integreren

$\boxed{A}$ Interpreteren van rationele functies en splitsen in partieelbreuken.

$$R(x) = \frac{P_1(x)}{P_2(x)}$$

1) Euclidische deling $R(x) = Q(x) + \frac{P_3(x)}{P_2(x)}$

$\deg P_3 < \deg P_2$

2) Zoek de nulpunten van P_2

- a is een reëel nulpunt van de orde n

$$P_2(x) = (x-a)^n P_4(x)$$

$$\Rightarrow \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$$

- (complex nulpunt)

$\alpha + i\beta$ $(\beta \neq 0)$ van de ~~n-de graad~~ orde n

$$\frac{A_1}{x-\alpha-i\beta} + \dots + \frac{A_n}{(x-\alpha-i\beta)^n} + \frac{B_1}{x-\alpha+i\beta} + \dots + \frac{B_n}{(x-\alpha+i\beta)^n}$$

$$\Rightarrow \frac{C_1x + D_1}{(x-\alpha)^2+\beta^2} + \dots + \frac{C_nx + D_n}{((x-\alpha)^2+\beta^2)^n}$$

$$\Rightarrow \frac{P_3}{P_2} = \frac{A_1}{(x-a)} + \dots + \frac{A_n}{(x-a)^n} + \frac{B_1}{(x-b)} + \dots + \frac{B_\ell}{(x-b)^\ell} + \frac{C_1x + D_1}{(x-\alpha)^2+\beta^2} \dots$$

Zie 7.3.5.

$\boxed{8}$ $\boxed{3}$ $\int \frac{dx}{x^4 - a^4}$

$$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$$
$$= (x - a)(x + a)(x^2 + a^2)$$

$$\frac{1}{x^4 - a^4} = \frac{A}{x-a} + \frac{B}{x+a} + \frac{Cx+D}{x^2+a^2}$$

'x' zonder een moet gelijk zijn aan 1

$$A(x+a)(x^2+a^2) + B(x-a)(x^2+a^2) + (Cx+D)(x^2-a^2) = 1$$

$$A(x^3+ax^2+a^2x+a^3) + B(x^3+a^2x-ax^2-a^3) + C(x^3-a^2x) + D(x^2-a^2) = 1$$

$$A + B + C = 0 \qquad Aa^2 + Ba^2 + Ca^2 = 0$$

$$Aa - Ba + D = 0 \qquad Aa^3 - Ba^3 - Da^2 = 1$$

$\Rightarrow$ Hieruit worden de waardes van A, B, C en D gevonden

$$A\ln(x-a) + B\ln(x+a) + \frac{C}{2}\int \frac{1}{x^2+a^2}\, d(x^2+a^2) + \frac{D}{a}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\, d\left(\frac{x}{a}\right)$$

$$\frac{C}{2}\ln(x^2+a^2) \qquad \frac{D}{a}\arctan\left(\frac{x}{a}\right)$$

Theorie deel $\boxed{B}$

Door substitutie ~~herlei~~ herleiden tot integreren van rationele functies.

1) $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{1/k}\right) dx = \int R\left(\frac{b-dt^k}{ct^k-a}, t\right) R_1(t)\,dt$

$\boxed{t = \left(\frac{ax+b}{cx+b}\right)^{\frac{1}{k}}} \Leftrightarrow x = \frac{b-dt^k}{ct^k-a}$

$dx = \left(\frac{b-dt^k}{ct^k-a}\right)' dt = R_1(t)\,dt$

2) $\int R(e^{ax})\,dx = \frac{1}{a}\int \frac{R(t)}{t}\,dt$

$\boxed{e^{ax} = t}$ $\quad ae^{ax}dx = dt \Leftrightarrow dx = \frac{dt}{at}$

3) $\int R(\sin x, \cos x)\,dx = 2\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{dt}{1+t^2}$

$\boxed{t = \tan\frac{x}{2}}$ $\quad x = 2\arctan t$, $\quad dx = \frac{2}{1+t^2}dt$

$\sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$

Oefeningen: $\boxed{8}$ $\boxed{4, 5, 6, 7, 9, 12, 13}$

$D = b^2 - 4ac$

$x_n = \frac{-b \pm \sqrt{D}}{2a}$

$\boxed{8}$ $\boxed{4}$ $\int \frac{x}{x^3+1}\,dx$

we weten dat -1 een nulpunt is:

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & 1 \\ -1 & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array} \rightarrow (x+1)(x^2-x+1)$$

$\boxed{x_1 = -1}$

nulpunten zoeken van (x^2-x+1)

$D = 1 - 4.1.1 = -3$

$x_2 = \frac{1+\sqrt{3}i}{2} = \frac{1}{2} + \frac{\sqrt{3}i}{2}$

$x_3 = \frac{1-\sqrt{3}i}{2} = \frac{1}{2} - \frac{\sqrt{3}i}{2}$

$(x-\alpha)^2 + \beta^2$

$\Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$ → bij complexe nulpunten kan het zo geschreven worden maar is eig. pas nodig bij oplossen van de integraal

$\rightsquigarrow \frac{x}{x^3+1} = \frac{ax+b}{(x-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + \frac{c}{x+1}$

$x = (a\cancel{x}\,x + b)(x+1) + c^*(x^2-x+1)$

$= a(x^2+x) + b(x+1) + c(x^2-x+1)$

$$\begin{aligned} a + c &= 0 \\ a + b - c &= 1 \\ b + c &= 0 \end{aligned} \Leftrightarrow \begin{cases} -c = a \\ a+b-c = 1 \\ c = -b \end{cases} \Leftrightarrow \begin{cases} -c = a \\ a+a+a = 1 \\ -c = b \end{cases}$$

$$\begin{cases} a = \frac{1}{3} \\ b = \frac{1}{3} \\ c = -\frac{1}{3} \end{cases}$$

$$\cancel{\Rightarrow \int \frac{x}{x^3+1}\,dx}$$

$$\Rightarrow \frac{x}{x^3+1} = \frac{ax+b}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{c}{x+1} = \frac{1}{3}\,\frac{x+1}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \frac{1}{3}\,\frac{1}{x+1}$$

1) $\frac{1}{3}\int \frac{dx}{x+1} = \frac{1}{3}\ln|x+1|$

2) $\frac{1}{3}\int \frac{x+1}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\,dx$

Stel: $\boxed{x - \frac{1}{2} = \frac{\sqrt{3}}{2}t}$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{3}\int \frac{\frac{\sqrt{3}}{2}t + \frac{3}{2}}{(\frac{\sqrt{3}}{2}t)^2 + (\frac{\sqrt{3}}{2})^2}\,dt = \frac{1}{3}\int \frac{t+\sqrt{3}}{t^2+1}\,dt$$

$$= \frac{1}{6}\int \frac{2t}{t^2+1}\,dt + \frac{1}{\sqrt{3}}\int \frac{dt}{t^2+1}$$

$$= \frac{1}{6}\ln(t^2+1) + \frac{1}{\sqrt{3}}\arctan t$$

$$= \frac{1}{6}\ln\left(\frac{4}{3}(x^2-x+1)\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$\Rightarrow \int \frac{x\,dx}{x^3+1} = \frac{1}{6}\ln\left(\frac{x^2-x+1}{(x+1)^2}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

2~~8~~7-10-16

$\boxed{5}$ $$\int \sqrt{\frac{a-x}{x-b}}\,dx = \int \sqrt{\frac{(a-b)-u}{u}}\,du \qquad (u = x-b)$$

$$\boxed{\text{stel: } \quad t = \sqrt{\frac{(a-b)-u}{u}} \Leftrightarrow u = \frac{a-b}{t^2+1}, \qquad du = \frac{-2(a-b)t\,dt}{(t^2+1)^2}}$$

$$= -2(a-b)\int \frac{t^2}{(1+t^2)^2}\,dt$$

$$\boxed{v = t \quad \rightarrow \quad w = \frac{-1}{2(t^2+1)}, \qquad dw = \frac{t\,dt}{(t^2+1)^2} = \frac{d(t^2+1)}{2(t^2+1)^2}}$$

$$= \frac{(a-b)t}{t^2+1} - (a-b)\arctan t$$

$$= \frac{(a-b)\sqrt{\frac{a-x}{x-b}}}{\frac{a-x}{x-b}+1} - (a-b)\arctan\left(\sqrt{\frac{a-x}{x-b}}\right)$$

$\boxed{6}$ $$\int \frac{1}{x^{1/2}+x^{1/6}}\,dx = \int \frac{1}{(x^{1/6})^3 + x^{1/6}}\,dx$$

$$\boxed{x^{1/6} = t \Leftrightarrow t^6 = x, \qquad dx = 6t^5\,dt}$$

$$= 6\int \frac{t^5\,dt}{t^3+t} = 6\int \frac{t^4\,dt}{t^2+1}$$

$$= 6\int \frac{t^4+1-1}{t^2+1}\,dt$$

$$= 6\int \frac{t^4-1}{t^2+1}\,dt + 6\int \frac{1}{1+t^2}\,dt$$

$$= 6\int (t^2-1)\,dt + 6\arctan t$$

$$= 2t^3 - 6t + 6\arctan t$$

$$= 2x^{1/2} - 6x^{1/6} + 6\arctan x^{1/6}$$

$$\cosh = \frac{e^x - e^{-x}}{2}$$

$\boxed{7}$ $$\int \frac{1}{\cosh x}dx = \int \frac{2dx}{e^x+e^{-x}} = \int \frac{2e^x dx}{e^{2x}+1} \begin{matrix}\to \text{maal } e^x \\ \to \text{maal } e^x\end{matrix}$$

$$\boxed{\begin{aligned} &\text{stel } e^x = t \\ &\Rightarrow e^x dx = dt \end{aligned}}$$

$$= \int \frac{2dt}{t^2+1} = 2\arctan t = 2\arctan e^x + C$$

$\boxed{9}$ $$\int \frac{dx}{\tan x + \sin x} = \int \frac{\cos x}{\sin x(1+\cos x)} dx$$

$$\boxed{\begin{aligned} t &= \tan\frac{x}{2} & \cos x &= \frac{1-t^2}{1+t^2} \\ dx &= \frac{2}{1+t^2}dt & \sin x &= \frac{2t}{1+t^2} \end{aligned}}$$

$$= \int \frac{1-t^2}{2t\left(\frac{2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \frac{1}{2}\int \frac{1}{t}dt - \frac{1}{2}\int t\,dt$$

$$= \frac{1}{2}\ln(t) - \frac{1}{4}t^2 = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + C$$

$\boxed{12}$ $$\int \frac{1}{\sin^3 x}dx = \overset{①}{\int \frac{\cos^2 x}{\sin^3 x}dx} + \overset{②}{\int \frac{1}{\sin x}dx} = ⊛$$

① $$\boxed{\begin{aligned} u &= \cos x & du &= -\sin x \\ dv &= \frac{\cos x\,dx}{\sin^3 x} = \frac{d\sin x}{\sin^3 x} & v &= \frac{-1}{2\sin^2 x} \end{aligned}}$$

$$① = \frac{-\cos x}{2\sin^2 x} - \frac{1}{2}\int \frac{1}{\sin x}dx$$

$$⊛ = \frac{-\cos x}{2\sin^2 x} + \frac{1}{2}\int \frac{1}{\sin x}dx$$

$\tan\frac{x}{2} = t$

$$⊛ = \frac{-\cos x}{2\sin^2 x} + \frac{1}{2}\int \frac{\cancel{1+t^2}}{2t} \cdot \frac{2dt}{\cancel{1+t^2}}$$

$$= \frac{-\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C$$

28-10-11

Theorie:

$\int R(x, \sqrt{ax^2+bx+c}\ dx \quad a \neq 0, \quad \Delta = b^2 - 4ac \neq 0)$

① b wegwerken: $x = t - \frac{b}{2a}$

$\Rightarrow ax^2 + bx + c = at^2 - \underbrace{\left(\frac{b^2-4ac}{4a}\right)}_{= d}$

② we onderscheiden 3 gevallen

(1) $\Delta > 0, \quad a > 0 \Rightarrow d > 0$

$t = \sqrt{\frac{d}{a}} \cosh u$

$dt = \sqrt{\frac{d}{a}} \sinh u$

$\sqrt{at^2 - d} = \sqrt{d} \sinh u$

$\int R(x, \sqrt{ax^2+bx+c})dx = \sqrt{\frac{d}{a}} \int R \sqrt{\frac{d}{a}} \cosh u - \frac{b}{2a} \sqrt{d}$

(2) $\Delta > 0$, ~~$0 < a$~~ $a < 0 \Rightarrow d < 0$

$t = \sqrt{\frac{d}{a}} \sin u$

(3) $\Delta < 0, a > 0 \Rightarrow d < 0 \quad t = \sqrt{-\frac{d}{a}} \sinh u$

$\boxed{8}$ $\boxed{18}$

$\boxed{9}$ $\boxed{14 \quad 2 \quad 3 \quad 5 \quad 9 \quad 12}$

$\boxed{8}$ $\boxed{18}$ $\displaystyle\int \frac{\sqrt{x^2+4x+5}}{2+x+\sqrt{x^2+4x+5}} dx$

$x = t - \frac{b}{2a} = t - 2$

$x^2 + 4x + 5 = at^2 - \frac{b^2-4ac}{4a}$

$d = \frac{b^2-4ac}{4a}$

$= \frac{16-20}{4} = -1$

... /

$\boxed{17}$ $\int \frac{x}{4-x^2+\sqrt{4-x^2}}\,dx$

$\boxed{x = t - \frac{b}{2a} = t}$

$= \int \frac{x\,dx}{\sqrt{4-x^2}\,(\sqrt{4-x^2}+1)}$

stel $\boxed{\sqrt{4-x^2} = t \Rightarrow \frac{-x}{\sqrt{4-x^2}}\,dx = dt}$

$= \int \frac{-dt}{t+1} = \ln|t+1| = -\ln(1 + \cancel{\ }\sqrt{4-x^2} + c)$

$\boxed{9}$ $\boxed{14}$ $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ $\quad a<0$, $b^2-4ac>0$

$x = t - \frac{b}{2a}$

$\int \frac{dt}{\sqrt{at^2-d}}$ met $d = \frac{b^2-4ac}{4a}$

$t = \sqrt{\frac{d}{a}}\sin u$

$= \frac{1}{\sqrt{-a}} \int \frac{\cos u}{\cos u}\,du = \frac{1}{\sqrt{-a}}\,u = \frac{1}{\sqrt{-a}}\,\mathrm{bgsin}\left(\sqrt{\frac{a}{d}}\left(x+\frac{b}{2a}\right)\right)$

$= \frac{1}{\sqrt{-a}}\,\mathrm{bgsin}\left(\frac{-2a-x}{\sqrt{b^2-4ac}}\right)$

$\boxed{2}$ $\int \frac{dx}{(1+x^2)^{5/2}}$

stel :
$$\boxed{\begin{aligned} \sinh t &= x \\ t &= \mathrm{bgsinh}\, x \\ dt &= \frac{dx}{\cosh t} \\ dx &= dt \cosh t \end{aligned}}$$

$$\int \frac{dt \cosh t}{(1+\sin^2 ht)^{5/2}} = \int \frac{dt \cosh t}{(1+\sin^2 ht)^2 \cdot \sqrt{1+\sinh^2 t}}$$

$$= \int \frac{\cancel{\cosh t}\, dt}{(\cosh^4 t)\cdot \cancel{(\cosh t)}}$$

$$= \int \frac{dt}{\cosh^4 t}$$

$$= \int \frac{\cosh^2 t - \sinh^2 t}{\cosh^4 t}\, dt$$

$$= \int \frac{dt}{\cosh^2 t} - \int \tanh^2 t \cdot \frac{dt}{\cosh^2 t}$$

$$= \tan t - \frac{\tanh^3 t}{3} = \frac{x}{\sqrt{1+x^2}} - \frac{1}{3}\left(\frac{x}{\sqrt{1+x}}\right)^2$$

3-11-16

10 | 2 3 5 7 9 11 12 14

10 | 2

$$\int \frac{x^3}{(a^2+x^2)}\,dx$$

~~nulpunten van a^2+x^2~~

~~$\frac{x^3}{(a^2+x^2)} = \frac{Ax+B}{(a^2+x^2)}$~~

$$= \int \frac{-2x}{(a^2+x^2)} \cdot \frac{-x^2}{2}\,dx = \frac{-x^2}{2}\frac{1}{(a^2+x^2)} + \int \frac{x}{a^2+x^2}$$

$$\boxed{\begin{array}{l} \text{stel } x^2 = t \\ 2x\,dx = dt \end{array}}$$

$$= \frac{-x^2}{2(a^2+x^2)} + \frac{1}{2}\int \frac{dt}{a^2+t} = \frac{-x^2+a^2-a^2}{2a^2+x^2} + \frac{1}{2}\ln(a^2+x^2)$$

$$= \frac{a^2}{2(x^2+a^2)} + \frac{1}{2}\ln(a^2+x^2) + C$$

10 | 3

$$\int \frac{\arcsin x}{(1-x^2)^{3/2}}\,dx =$$

$$\boxed{\begin{array}{ll} u = \arcsin x & du = \frac{1}{\sqrt{1-x^2}}\,dx \\ dv = \frac{1}{(1-x^2)^{3/2}} & v \overset{(*)}{=} \frac{x}{\sqrt{1-x^2}} \end{array}}$$

$$= \frac{x\arcsin x}{\sqrt{1-x^2}} - \int \frac{x}{1-x^2}\,dx = \frac{1}{2x}\frac{x\,d(1-x^2)}{1-x^2}$$

$$2x = d(1-x^2)$$

$$= \frac{x\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2}\ln(1-x^2)$$

$$(*) \int \frac{1}{(1-x^2)^{3/2}}\,dx = \int \frac{1}{\cos^2 t}\,dt$$

$$= \tan T$$

$$= \frac{x}{\sqrt{1-x^2}}$$

$$\boxed{x = \sin t}$$

$$dx = \cos t\,dt$$

$$\int \frac{1}{(1-\sin^2 t)^{3/2}}\cos t\,dt = \int \frac{1}{(\cos^2 t)^{3/2}}\cos t\,dt = \int \frac{1}{\cos^2 t}$$

$$= \cancel{\tan T}$$

$$= \cancel{\tan(\arcsin x)}$$

$$= \tan T$$

$$= ?$$

$$= \frac{x}{\sqrt{1-x^2}}$$

$\boxed{5}$ $\displaystyle\int \frac{\sin 2x}{\cos 3x}\,dx = \int \frac{2\sin x\cos x}{\cos^2 x\cos x - 2\sin^2 x\cos x}\,dx$

$$= \int \frac{2\sin x\,dx}{\cos^2 x - 2\sin^2 x\cdot 1} = \int \frac{2\sin x}{4\cos^2 - 3}\,dx$$

$\boxed{\text{Stel } \frac{2\cos x}{\sqrt{3}} = t \qquad \frac{-2\sin x}{\sqrt{3}}\,dx = dt}$

$$= \sqrt{3}\int \frac{-dt}{3(t^2-1)} = \frac{1}{\sqrt{3}}\int \frac{dt}{1-t^2} = \frac{1}{\sqrt{3}}\operatorname{argtanh} t$$

$$= \frac{1}{\sqrt{3}}\operatorname{argtanh}\left(\frac{2\cos x}{\sqrt{3}}\right) + C$$

$\boxed{7}$ $\displaystyle\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{1}{b^2}\int \frac{1}{\left(\frac{a^2}{b^2}\frac{\sin^2 x}{\cos^2 x} + 1\right)\cos^2 x}\,dx$

stel : $\frac{a}{b}\tan x = t$

$$= \frac{b}{b^2 a}\int \frac{1}{1+t^2}\,dt$$

$$= \frac{1}{ab}\arctan\left(\frac{a}{b}\tan x\right)$$

$\boxed{9}$ $\displaystyle\int \sin^2(\sqrt{x})\,dx$

$\boxed{\begin{array}{l}\text{Stel } \sqrt{x} = t \\ dx = 2t\,dt\end{array}}$

$$= \int 2t\sin^2 t\,dt$$
$$= \int t(1-\cos 2t)\,dt$$
$$= \frac{t^2}{2} - \int t\cos 2t\,dt$$
$$= \frac{t^2}{2} - \frac{t\sin 2t}{2} + \int \frac{\sin 2t}{2}\,dt$$
$$= \frac{x}{2} - \frac{\sqrt{x}}{2}\sin(2\sqrt{x}) - \frac{\cos(2\sqrt{x})}{4} + C$$

7-11-16

$\boxed{10}$ $\boxed{22 \quad 23}$

$\boxed{14}$ $\boxed{1 \quad 3 \quad 5 \quad 7}$

$\boxed{10}$ $\boxed{22}$ $\int \sqrt{2 - \frac{1}{\cos^2 x}}\, dx$

* $\boxed{\begin{array}{l} \tan x = t \\ t = \sin u \end{array}}$

$$= \int \frac{\sqrt{1-t^2}}{1+t^2}\, dt = \int \frac{\cos^2 u}{1+\sin^2 u}$$

$$= \int \frac{1}{1+\sin^2 u}\, du - \int \frac{\sin^2 u}{1+\sin^2 u}\, du$$

$$= 2\int \frac{1}{1+\sin^2 u}\, du - u$$

$$= 2\int \frac{1}{2\tan^2 u+1} \frac{du}{\cos^2 u} - u$$

$$= \sqrt{2} \arctan(\sqrt{2} \tan u) - u \quad \text{met } u = \arcsin(\tan x)$$

$\boxed{23}$ $\int \sin x \ln(\sin x)\, dx = -\cos x \ln(\sin x) + \int \frac{\cos^2 x}{\sin x} dx$

stel : $\boxed{\begin{array}{l} \cos x = t \\ x = \arccos t = \frac{\pi}{2} - \arcsin t \\ dx = \frac{-dt}{\sqrt{1-t^2}} \end{array}}$

$$= -\cos x \ln(\sin x) + \int \frac{t^2}{\sqrt{1-t^2}} \cdot \frac{-dt}{\sqrt{1-t^2}} = -\cos x \ln(\sin x) - \int \frac{t^2 dt}{1-t^2}$$

$$= -\cos x \ln(\sin x) + t - \int \frac{dt}{1-t^2}$$

$$= -\cos x \ln(\sin x) + \cos x - \operatorname{argtanh}(\cos x) + C$$

Differentiaalvergelijkingen

zoek $y(x)$:

$g(y(x))\, y'(x) = f(x)$ soms beginvoorwaarde: $y(x_0) = y_0$

$\int g(y(x))\, y'(x)\, dx = \int f(x)\, dx$

$\Leftrightarrow G(y(x)) = F(x) + C$ G = primitieve van g
F = primitieve van f

14 1 $y' = x^2 y^3$ $y(1) = 3$

$\frac{1}{y^3} y' = x^2$

$\int \frac{1}{y^3} \cancel{y}\, dy = \int \frac{1}{y^3} y'\, dx = \int x^2\, dx + C$

$\Leftrightarrow -\frac{1}{2y(1)^2} = \frac{-1}{2\cdot 9} = \frac{1}{3} + C \Leftrightarrow C = \frac{-7}{18}$

$y = \pm\sqrt{\frac{18}{14 - 12x^2}} \overset{y(1)=3}{\Rightarrow} y = 3\sqrt{\frac{1}{7 - \cancel{8}6x^2}}$

3 $y' = e^{2x+5y}$ $y(0) = -1$

$\frac{e^{-5y}}{-5} = \int e^{-5y}\, dy = \int e^{2x}\, dx = \frac{e^{2x}}{2} + C$

$\Rightarrow e^{-5y} = \frac{-5}{2} e^{2x} + c \Rightarrow -5y = \ln\left(\frac{-5}{2} e^{2x} + c\right)$

$\cancel{y(0) =}$ $\cancel{\ln}$ $y = \frac{-\ln\left(\frac{-5}{2} e^{2x} + c\right)}{5}$

$y(0) = \frac{-\ln\left(\frac{-5}{2} + c\right)}{5} = -1 \Rightarrow \frac{-5}{2} + c = e^5$

$c = e^5 + \frac{5}{2}$

$\Rightarrow y = \frac{-\ln\left(\frac{-5}{2} e^{2x} + e^5 + \frac{5}{2}\right)}{5}$

$\boxed{5}$ $y' = (7y\ln x)/3$ met $y(1) = e$

$$\boxed{\begin{array}{ll} u = \ln x & du = \frac{1}{x}dx \\ dv = dx & v = x \end{array}}$$

$\frac{1}{y} y' = \frac{7}{3} \ln x$

$\int \frac{1}{y} dy = \int \frac{1}{y} y' dx = \frac{7}{3} \int \ln x dx$

$\ln y = \frac{7}{3}(x\ln x - x) + c$

$\Rightarrow C = \frac{10}{3}$

$y = \frac{e^{10/3} \, x^{7/3 x}}{e^{7/3 x}}$

$\boxed{7}$ $y' = 4y^2 - 1$ $\qquad y(0) = -\frac{3}{2}$

$\frac{y'}{4y^2-1} = 1 \quad \Rightarrow \quad x + C = \int dx = \int \frac{dy}{4y^2-1} = \frac{-1}{2} \operatorname{argtanh} 2y$

$\Rightarrow 2y = \tanh(-2x + c)$

$y(0) = \frac{1}{2} \frac{c-1}{c+1} = \frac{-3}{2} \Rightarrow c = \frac{-1}{2}$

$y = \frac{-1}{2} \frac{e^{-4x + cte} - 1}{e^{-4x+c} + 1} = \frac{1}{2} \frac{cte\, e^{-4x} - 1}{cte\, e^{-4x} + 1}$

$y = \ldots$

29-11-16

Hoofdstuk 10

3) $F(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \quad (-1 < k < 1)$

→ nemen x als vaste constante

$$\int_0^{\pi/2} \frac{1\,dx}{\sqrt{1-k^2\sin^2 x}} = \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2\sin^2 x \ \frac{1\cdot 3}{2\cdot 4}k^4\sin^4 x + \ \dots \right) dx$$

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \left[\frac{-\sin^{2n-1} x\cos x}{2n}\right]_0^{\pi/2} + \frac{2n-1}{2n}\int_0^{\pi/2} \sin^{2n-2} x\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{\pi}{2}$$

$$\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\pi}{2}\left(1 + \sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 k^{2n}\right)$$

~~Theorie~~

4) a) $\sqrt{\frac{1+x}{1-x}}$ (tot en met x^2)

$$= (1+x)^{1/2}\cdot(1-x)^{-1/2}$$

$$= \left(1 + \frac{x}{2} - \frac{x^2}{8} + O(x^3)\right)\left(1 + \frac{x}{2} + \frac{3x^2}{8} + O(x^3)\right)$$

$$= 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{x}{2} + \frac{x^2}{4} - \frac{x^2}{8} + O(x^3)$$

$$= 1 + x + \frac{x^2}{2} + O(x^3)$$

b) $\cosh x \sin x$ (tot en met x^3)

$$= \left(1 + \frac{x^2}{2} + O(x^4)\right)\left(x - \frac{x^3}{6} + O(x^4)\right)$$

$$= x - \frac{x^3}{6} + \frac{x^3}{2} + O(x^4) = x + \frac{x^3}{3} + O(x^4)$$

c) $\tan x$ tot en met x^5

~~$= \frac{\sin x}{\cos x}$~~

~~$= \dfrac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - O(x^6)}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - O(x^5)}$~~

$$= a_0 + a_1 x + a_2 x^2 + \dots$$

$$\Rightarrow (a_0 + a_1 x + a_2 x^2 + \dots)\cdot\cos x = \sin x$$

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

⇒ Stelsel maken

$a_0 = 0$

$a_1 = 1$

$a_2 = -\frac{1}{2}a_0 = 0$

$-\frac{1}{6} = a_3 - \frac{a_1}{2}$

$\frac{a_0}{4!} - \frac{a_2}{2} + a_4 = 0$

$\frac{1}{5!} = a_5 - \frac{a_3}{2} + \frac{a_1}{4!}$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^6)$$

Fourierreeksen

1.1) $f(x) = x$ ontwikkelen t.o.v. $[-\pi, \pi]$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

met $a_n, b_n \in \mathbb{R}$.
De reekssom is 2π periodieke functie van x.
(waar ze convergeert (?))

Stel $f : [-\pi, \pi] \to \mathbb{R}$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n = 1, 2, 3 \ldots$$

Als: f = even $\Rightarrow (f(-x) = f(x)) \Rightarrow b_n = 0$
f = oneven $(f(x) = -f(-x)) \Rightarrow a_n = 0$

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

- wanneer convergeert de fourierreeks van f?
- als ze convergeert in x, wanneer convergeert ze naar f(x)?

$f : [-\pi, \pi] \to \mathbb{R}$ stuksgewijs continu op $[-\pi, \pi]$
f^{π} = functie die ontstaat door f $(D =]-\pi, \pi[)$ met periode 2π voort te zetten. $\Rightarrow f^{\pi} : \mathbb{R} \to \mathbb{R}$

$\Rightarrow$ 2π-periodiek en stuksgewijs continu op $[-\pi, \pi]$

$f : \mathbb{R} \to \mathbb{R}$ stuksgewijs continu op $[a, b]$
$\exists\, a = x_0 < x_1 < \ldots < x_i = b$ zodat

$f|_{]x_i, x_{i+1}[}$ $\to$ metrisch continue functie op $]x_i, x_{i+1}[$

1.1) $f(x) = x$ tov $[-\pi, \pi]$

$u = x \qquad dv = \sin nx$
$du = dx \qquad v = \frac{-\cos nx}{n}$

$\to$ oneven $\Rightarrow a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx$$

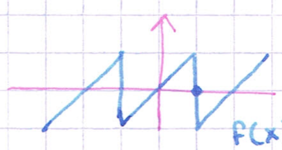
$$= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx = \frac{2}{\pi}\left(\left[\frac{x \cdot -\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx\,dx\right)$$

oneven · oneven = even

$$b_n = \frac{-2\cos n\pi}{n} + \frac{1}{n}\cdot 0 = \frac{-2(-1)^n}{n}$$

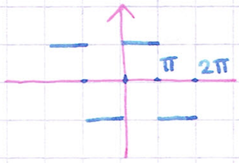
$$\Rightarrow f \sim -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx$$

$$= -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx = \begin{cases} x & x \in]-\pi, \pi[\\ 0 & x = \pm\pi \end{cases}$$



$f(x) = 0 \to x = \pm\pi$

③ $f(x) = \begin{cases} -1 & x \in](2k-1)\pi, 2k\pi[\\ 1 & x \in]2k\pi, (2k+1)\pi[\end{cases}$



ontwikkelen t.o.v. $]-\pi, \pi[$

oneven: $a_n = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$

oneven maal oneven = even

$= \frac{2}{\pi} \int_0^{\pi} (+1) \sin nx \, dx$

↳ van 0 tot π is het +1

$= \frac{2}{\pi} \left[\frac{-\cos nx}{n} \right]_0^{\pi} = \frac{2}{\pi} \left[\frac{1}{n} - \frac{\cos n\pi}{n} \right]$

(1 - -1)(oneven)

$= \frac{2}{\pi n} (1 - \cos n\pi) = \begin{cases} 0 & \text{als } n \text{ oneven} \\ \frac{4}{\pi n} & \text{als } n \text{ even} \end{cases}$

(teken grafiek)

$f \sim \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin((2n+1)x) = \begin{cases} f(x) & x \in]-\pi, 0[\cup]0, \pi[\\ 0 & x = -\pi, 0, \pi \end{cases}$

→ Voor n = even is het 0, we nemen enkel oneven termen

④ $f(x) = x^2$

$=$ even $\Rightarrow b_n = 0$

$a_0 = \frac{1}{\pi} 2 \int_0^{\pi} f(x) dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$

mag als het even is

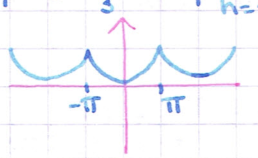
$a_n = \frac{1}{\pi} 2 \int_0^{\pi} f(x) \cos nx \, dx$

$= \frac{1}{\pi} 2 \int_0^{\pi} x^2 \cos nx \, dx$ $\quad u = x^2 \quad du = 2x\,dx \quad dv = \cos nx \quad v = \frac{\sin nx}{n}$

...

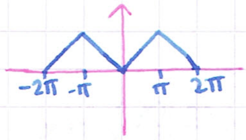
$= \frac{-4}{\pi n} \left(\frac{-x \cos nx}{n} \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right) = \frac{4(-1)^n}{n^2}$

$f \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} = f(x) \; \forall x \in [-\pi, \pi]$



② $f(x) = \begin{cases} x & x \in [0, \pi] \\ 2\pi - x & x \in [\pi, 2\pi] \end{cases}$

ontwikkel in een reeks van cosinussen



enkel cosinussen dus we breiden de functie even uit.

g op $[-2\pi, 2\pi] =$ even $\rightarrow$ $g_{|[\cancel{-2\pi}\ 0, 2\pi]} = f$

$a_0 = \pi$

$\cancel{a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{nx\pi}{L}\,dx$

$a_n = \frac{1}{2\pi}\,2\int_0^{2\pi} x\cos\frac{n\pi x}{2\pi} + \frac{1}{2\pi}\,2\int_\pi^{2\pi}(2\pi - x)\cos\frac{nx}{2}\,dx$

maal 2 want het is even dus men kan splitsen.

$\Rightarrow \int_\pi^{2\pi}(2\pi - x)\cos\frac{nx}{2}\,dx$ $\quad u = x \quad du = dx \quad dv = \cos\frac{n}{2}x \quad v = \frac{\sin\frac{n}{2}x}{\frac{n}{2}}$

$= x\sin\frac{n}{2}x\cdot\frac{2}{n} - \frac{2}{n}\int\sin\frac{n}{2}x\,dx$

$= x\sin\frac{n}{2}x\cdot\frac{2}{n} + \frac{2}{n}\cos\frac{n}{2}x\cdot\frac{2}{n}$

$= \frac{2x}{n}\sin\frac{n}{2}x + \frac{4\cos\frac{nx}{2}}{n^2}$

$a_n = \frac{1\cdot 2}{n\pi}\left[x\sin\frac{nx}{2} + \frac{2}{n}\cos\frac{nx}{2}\right]_0^\pi - \frac{1}{\pi n}\left[x\sin\frac{nx}{2} + \cos\frac{nx}{2}\cdot\frac{2}{n}\right]_\pi^{2\pi}$
$+ 2\left[\sin\frac{nx}{2}\cdot\frac{2}{n}\right]_\pi^{2\pi}$

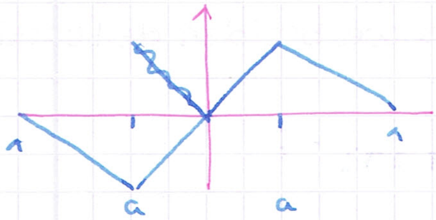
$= \begin{cases} 0 & \text{als } n \text{ oneven} \\ \frac{8}{\pi n^2}\left((-1)^{n/2} - 1\right) & \text{als } n \text{ even} \end{cases}$

$f(x) \sim \frac{\pi}{2} - \frac{16}{\pi}\left(\frac{\cos x}{4} + \frac{\cos 3x}{36} + \frac{\cos 5x}{100} + \dots\right) \quad x \in [0, 2\pi]$

er zijn geen sprongpunten

④ moet oneven zijn over $[0,1]$

$$f(x)\begin{cases}(1-a)x & \forall x\in[0,a] \\ a(1-x) & \forall x\in[a,1]\end{cases} \qquad 0<a<1$$



$a_n = 0$

$$b_n = \frac{1}{\pi}\cdot 2\int_0^1 f(x)\sin\left(\frac{2\pi x}{1}\right)dx$$

$$= \int_0^a (1-a)x\sin(n\pi x)dx + a\int_a^1 \underbrace{(1-x)}_{t=1-x}\sin(n\pi x)dx$$

$(\sin n\pi(1-x) = -\cos n\pi \cdot \sin(n\pi x))$

$$= 2(1-a)\int_0^a x\sin(n\pi x)dx - a\cos n\pi \int_0^{1-a} \cancel{(1-x)}\sin(n\pi x)dx$$

grenzen aanpassen? $t=1-1$, $t=1-a$ → omdraaien en '−' voor zetten

$u=x \qquad dv=\sin n\pi x$

$du=dx \qquad v=-\cos n\pi x\cdot\frac{1}{n\pi}$

$$= 2(1-a)\left(\left[-x\frac{\cos n x}{n\pi}\right]_0^a + \frac{1}{n\pi}\int_0^a \cos n\pi\, dx\right)$$

$$- 2a\cos(n\pi)\left(\left[\frac{-x\cos n\pi x}{n\pi}\right]_0^{1-a} + \int_0^{1-a}\cos n\pi x\, dx\right.$$

$$= 2(1-a)\left(\left(\frac{-a\cos n\pi a}{n\pi}\right) + \frac{1}{(n\pi)^2}\sin n\pi a\right]$$

$$- 2a\cos n\pi\left(\frac{-(1-a)\cos(n\pi(1-a))}{n\pi} + \frac{1}{(n\pi)^2}\sin n\pi a\right]$$

$$= \frac{2\sin n\pi a}{(n\pi)^2}$$

$$g \sim \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi a)}{n^2}\sin(n\pi x) = f(x) \quad \forall x\in[0,1]$$

8-12-16

Hoofdstuk 11

6, 8, 9, 11 (2), 12

6) ~~cos~~ $\cos \alpha x$ $(\alpha \in \mathbb{Z})$ t.o.v. $[0, \pi]$

→ uitbreiden tot $[-\pi, \pi]$

Even functie:

$\cancel{a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos \alpha x \, dx = \frac{1}{\pi} \frac{1}{\alpha} \int_{-\pi}^{\pi} \cos \alpha x \, d\alpha x}$

$b_n = 0$ $\cancel{= \frac{1}{\pi\alpha} [\sin \alpha x]_{-\pi}^{\pi}}$

$$\frac{a_0}{2} = \frac{1}{2\pi\alpha} \int_{-\pi}^{\pi} \cos \alpha x \, dx\alpha = \frac{1}{\pi\alpha} \sin \pi\alpha$$ $(\sin \alpha\pi - \sin(-\alpha\pi) = \sin \alpha\pi + \sin\alpha\pi)$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cos \alpha x \cos nx \, dx$$ (2 keer partiële integratie) simpson!

$$\int_0^{\pi} \cos \alpha x \cos nx \, dx = \frac{-\alpha}{n^2} \int_0^{\pi} \sin \alpha x \sin nx \, dx$$
$$= \frac{-\alpha}{n^2} \cos n\pi \sin \alpha\pi$$
$$+ \frac{\alpha^2}{n^2} \int_0^{\pi} \cos \alpha x \cos nx \, dx$$

$u = \cos \alpha x$

$du = -\alpha \sin \alpha x \, dx$

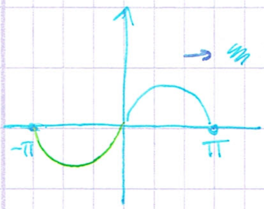
$dv = \cos nx$ $\quad \Rightarrow \left(1 - \frac{\alpha^2}{n^2}\right) \int_0^{\pi} \cos \alpha x \cos nx \, dx$

$v = \frac{1}{n} \sin nx$ $\quad = \frac{-\alpha}{n^2} \cos n\pi \sin \alpha\pi$

$$f \sim \frac{1}{\pi\alpha} \sin \alpha\pi + \frac{2\alpha}{\pi} \sin \alpha\pi \sum_{n=1} \frac{(-1)^n}{\alpha^2 - n^2} \cos nx = \cos \alpha x \quad \forall x \in [-\pi, \pi]$$

2) $$\frac{1}{\tan \alpha\pi} = \frac{1}{\sin \alpha\pi} \left(\frac{\sin \alpha\pi}{\pi\alpha} + \frac{2\alpha \sin \alpha\pi}{\pi} \sum \frac{(-1)^n}{\alpha^2 - n^2} \cos n\pi \right) = \frac{1}{\pi\alpha} + \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{1}{\alpha^2 - n^2}$$

8) (1) $x(\pi - x)$ $(0 < x < \pi)$ in een reeks die enkel sinussen bevat



→ gedefinieerd, enkel sinussen → oneven

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx \quad = \frac{2}{\pi}\int_0^\pi (x(\pi - x))\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \pi x \sin nx\,dx - \frac{2}{\pi}\int_0^\pi x^2 \sin nx\,dx$$

$$= 2\left[-\frac{x\cos nx}{n}\right]_0^\pi + \frac{2}{n}\int_0^\pi \cos nx\,dx + \frac{2}{\pi n}\left[x^2\cos nx\right]_0^\pi - \frac{4}{\pi n}\int_0^\pi x\cos nx\,dx$$

$$= \frac{2\pi}{n}\cos n\pi + \frac{2}{n^2}\left[\sin nx\right]_0^\pi + \frac{2\pi}{n}\cos n\pi - \frac{4}{\pi n^2}\left[x\sin nx\right]_0^\pi - \frac{4}{\pi n^2}\left[\frac{\cos nx}{n}\right]_0^\pi$$

$$= \frac{-4}{\pi n^3}\left((-1)^n - 1\right)$$

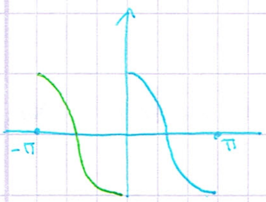
$$\Rightarrow f(x) \sim \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{27} + \dots\right) \quad x \in [0, \pi]$$

(3) bereken de reeksom $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \dots$

$$= \sin\frac{\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3^3} + \frac{\sin\frac{5\pi}{2}}{5^3} + \dots = \frac{\pi}{8}\cdot f(\pi) = \frac{\pi^3}{32}$$

$$\to \sin\left(k\pi + \frac{\pi}{2}\right) = (-1)^k$$

9) $f(x) := \cos x$ $(0 < x < \pi)$ in een reeks van sinussen



$$b_n = \frac{2}{\pi}\int_0^\pi \cos x \sin nx\,dx \quad = \frac{(-1)^n + 1}{n} - \frac{1}{n}\int_0^\pi \sin x \cos nx\,dx$$

(2x partiële integratie)

$$= \frac{(-1)^n + 1}{n} + \frac{1}{n^2}\int_0^\pi \cos x \sin nx\,dx$$

$$\Rightarrow \int_0^\pi \cos x \sin nx\,dx = \frac{n((-1)^n + 1)}{n^2 - 1}$$

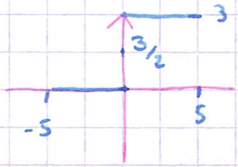
$$f \sim \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2 - 1}\sin(2nx)$$

$$= \begin{cases} f(x) & x \in]0, \pi[\\ 0 & x \in 0, \pi \end{cases}$$

$$= \frac{2n(1+1)}{(2n)^2 + 1}$$ (enkel voor even, oneven = 0)

$$= \frac{4n}{4n^2 + 1}$$

5) $f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$ tov $[-5,5]$



in de sprongpuntenconvergeert de reeks naar $\frac{3}{2}$

noch even, nog oneven

$$a_0 = \frac{1}{5}\left(\int_{-5}^{0} 0\, dx + \int_0^5 3\, dx\right) = \frac{1}{5}[3x]_0^5 = 3$$

$$a_n = \frac{1}{5}\left(\int_0^5 3\cos\frac{n\pi x}{5}\right) dx$$

$$= \frac{3}{\cancel{5}}\left[\sin\frac{n\pi x}{5}\right]_0^5 \frac{\cancel{5}}{n\pi} = \frac{3}{2n\pi}\cdot 0 - \frac{3}{2\pi}\cdot 0$$

$$b_n = \frac{1}{5}\int_0^5 3\sin\frac{n\pi x}{5}\, dx$$

$$= \frac{3}{5}\frac{5}{n\pi}\left[-\cos\frac{n\pi x}{5}\right]_0^5 = \frac{3}{n\pi}[-\cos n\pi + \cos 0]$$

$$= \frac{3}{n\pi} - (-1)^n + \frac{3}{n\pi} = \frac{3}{n\pi}(1 - \cos n\pi)$$

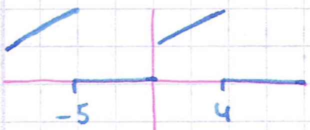
= 0 (n even)

$= \frac{6}{\pi n}$ (n oneven)

$$f \sim \frac{3}{2} + \frac{6}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi x}{5}\right) = \begin{cases} f(x) & x \in]-5,0[\cup]0,5[\\ \frac{3}{2} \text{ als } x = -5, 0, 5 \end{cases}$$

(blad 8/12/16)

(11) 2) $f(x) = \begin{cases} 2+3x & 0 \leq x < 4 \\ 0 & -4 \leq x < 0 \end{cases}$ του $[-4, 4]$



$$\frac{a_0}{2} = \frac{1}{2\cdot 4}\int_{-4}^{4} f(x)\,dx = \frac{1}{8}\int_0^4 (2\cancel{x}+3x)\,dx = 4$$

$\circledast$ $u = x$ $\quad dv = \cos\left(\frac{n\pi x}{4}\right)$

$du = dx$ $\quad v = \frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right)$

$$a_n = \frac{1}{4}\int_0^4 (2+3x)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\int_0^4 \cos\left(\frac{2\pi x}{4}\right)dx + \frac{3}{4}\int_0 x\cos\left(\frac{n\pi x}{4}\right)dx$$

$$\overset{\circledast}{=} \frac{-3\cdot 4}{\cancel{4}\,n\pi}\int_0^4 \sin\left(\frac{n\pi x}{4}\right)dx = \frac{12}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{2}\int_0^4 \sin\left(\frac{n\pi x}{4}\right)dx = \frac{3}{4}\int_0^4 x\sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) + \frac{3}{4}\left(\left[\frac{-4x}{n\pi}\cos\left(\frac{n\pi x}{4}\right)\right]_0^4 + \frac{4}{n\pi}\int_0^4 \cos\left(\frac{n\pi x}{4}\right)dx\right)$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) - \frac{12}{n\pi}(-1)^n = \frac{2}{n\pi}\left(1-7(-1)^n\right)$$

$$f \sim 4 + \sum_{n=1}^{\infty} \frac{12}{(n\pi)^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{4}\right) + \frac{2}{n\pi}\left(1-7(-1)^n\right)\sin\left(\frac{n\pi x}{4}\right)$$

$$= \begin{cases} f(x) & x \in]-4,0[\, \cup\,]0,4[\\ 1 & x = 0 \\ 7 & x - 4, 4 \end{cases}$$

(13) 1) $f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ x^2 & 0 \leq x \leq \pi \end{cases}$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{1}{\pi}\left[\frac{x^2\sin nx}{n}\right]_0^{\pi} + \left[\frac{2x}{\pi n^2}\cos nx\right]_0^{\pi} - \left[\frac{2}{\pi n^3}\sin nx\right]_0^{\pi} = (-1)^n\frac{2}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2\sin nx\,dx = -\frac{1}{\pi}\left[x^2\frac{\cos nx}{n}\right]_0^{\pi} + \left[\frac{2x}{\pi n^2}\sin nx\right]_0^{\pi} + \left[\frac{2}{\pi n^3}\cos nx\right]_0^{\pi} = -\frac{\pi}{n}(-1)^n + \frac{2}{\pi n^3}\left((-1)^n - 1\right)$$

$$f \sim \frac{\pi^2}{6} + \sum_{n\geq 1}\left(\frac{2(-1)^n}{n^2}\cos nx + \left(\frac{2}{\pi n^3}\left((-1)^n - 1\right) - \frac{\pi}{n}(-1)^n\right)\right)$$

Lineaire differentiaalvglingen

① 1e-orde dif. vgl

$$y' + a(x)y = R(x) \qquad (*)$$

$a, R : U \to \mathbb{R}$ ($U \subseteq \mathbb{R}$ open) gegeven continue functies en y de onbekende functie.

Stelling: Alle oplossingen van $(*)$ wordt gegeven door

$$y = e^{-\int a}\left(c + \int R e^{\int a}\right) \quad c \in \mathbb{R} \text{ willekeurig}$$

② 2-e-orde

$$y'' + a(x)y' + b(x) = R(x) \qquad (**)$$

$a, b, R \to$ gegeven continue functies en y de onbekende

Als $\quad y'' + a(x)y' + b(x) = 0 \qquad (***)$

noemen we de vgl homogeen

2 oplossingen φ_1, φ_2 van $(***)$ wordt onafhankelijk genoemd als $\nexists c \in \mathbb{R}$ $\varphi_1 = c\varphi_2$ of $c\varphi_1 = \varphi_2$

Stelling: ∘ Zij φ_1, φ_2 2 onafhankelijke oplossingen van $(***)$
=> Alle oplossingen van $(**)$ van de vorm

$$\underbrace{c_1\varphi_1 + c_2\varphi_2}_{=\text{ algemene oplossing}}, \quad c_1, c_2 \in \mathbb{R} \text{ willekeurig}$$

∘ Als φ_0 één oplossing van de niet-homogene vgl $(**)$
Alle oplossingen van $(**)$ gegeven door

$$c_1\varphi_1 + c_2\varphi_2 + \varphi_0 \qquad c_1, c_2 \in \mathbb{R}$$

φ_0 wordt een particuliere oplossing genoemd

dus a) alg oplossing v.d. homogene vgl
b) één particuliere oplossing van $(***)$

③ 2^e-orde met constante coëfficiënten

$y'' + py' + qy = R \quad (+) \qquad p, q \in \mathbb{R}$

a) Homogene vgl : $y'' + py' + qy = 0$
Karakteristieke veelterm : $P(x) := x^2 + px + q = 0$

! Stelling : ⊙ $\lambda_1, \lambda_2 \rightarrow$ 2 verschillende wortels van P
$\Rightarrow e^{\lambda_1 x}, e^{\lambda_2 x}$ onafhankelijke oplossingen

⊙ λ_0 dubbelwortel van $p \Rightarrow e^{\lambda_0 x}, xe^{\lambda_0 x}$ 2 onafhankelijke opl.
⊙ $(a \pm ib)$ complexe wortels $\Rightarrow e^{ax}\cos bx, e^{ax}\sin bx$

b) particuliere oplossing
3 methoden: - onbepaalde coëf (vandaag)
- verlaging van de orde
- operatormethode

$y'' + py' + qy = e^{ax}(C(x)\cos bx + S(x)\sin bx)$

$a, b \in \mathbb{R}$, C, S = polynomen (graad $\leq N$)

- $a + ib$ geen wortel is van P $\Rightarrow \exists$ ~~part~~ particuliere opl.
  $\varphi_0 = e^{ax}(C_0(x)\cos bx + S_0(x)\sin bx)$
  deg $C_0, S_0 \leq N$

- $a + ib$ enkelvoudige wortel ~~van~~ $\Rightarrow \exists$ particuliere opl.
  $\varphi_0(x) = xe^{ax}(C_0(x)\cos bx + S_0(x)\sin bx)$

- $a + ib$ dubbele wortel $(b = 0)$
  $\Rightarrow y'' - 2ay' + a^2 y = e^{ax} C(x)$

  $\varphi_0 = e^{ax} C_0(x)$
  $C_0(x) = \iint C(x)$

4.1) $y'' + 4y' - 5y = x^2$

a) <u>alg opl van de homogene:</u>

$x^2 + 4x - 5 = (x+5)(x-1)$

alg opl: $c_1 e^{\overset{\uparrow}{-5x}} + c_2 e^{\overset{\uparrow}{x}}$, $c_1, c_2 \in \mathbb{R}$

$\lambda_1 \qquad \lambda_2$

b) <u>particuliere opls van niet-homogene:</u>

"$a = b = 0$ "
$\Rightarrow \exists$ part opl van de vorm
$y(x) = \alpha x^2 + \beta x + \gamma$

$2\alpha + 4(2\alpha x + \beta) - 5(\alpha x^2 + \beta x + \gamma) = x^2$

$\Leftrightarrow (-5\alpha)x^2 + (8\alpha - 5\beta)x + (2\alpha + 4\beta - 5\gamma) = x^2$

$-5\alpha = 1$
$8\alpha - 5\beta = 0 \quad \Rightarrow \quad \alpha = \frac{-1}{5}$
$2\alpha + 4\beta - 5\gamma = 0 \quad\quad \beta = \frac{-8}{25}$
$\gamma = \frac{-42}{125}$

c) alg oplossing van de niet homogene vgl = alg opl hom + alg opl part.

$y(x) = c_1 e^{-5x} + c_2 e^x - \frac{x^2}{5} - \frac{8x}{25} - \frac{42}{125}$

1.1) $\begin{cases} x^2 y' + 2xy - 1 = 0 \\ y(x_0) = y_0 \end{cases}$ over $\mathbb{R}^+$
$y' + a(x)y = R(x)$

$\Leftrightarrow \begin{cases} y' + \frac{2}{x} y = \frac{1}{x^2} \\ y(x_0) = y_0 \end{cases}$ $\quad a(x) = \frac{-2}{x}$, $R(x) = \frac{1}{x^2}$

$y(x) = e^{-\int a}\left(c + \int R e^{\int a}\right)$

$= e^{-\ln x^2}\left(c + \int \frac{1}{x^2} e^{\ln x^2}\right)$

$= \frac{1}{x^2}(c + x)$, $c \in \mathbb{R}$ (algemene oplossing)

c bepalen door de beginvoorwaarde $y(x_0) = y_0$ => c vinden

$y_0 = y(x_0) = \frac{1}{x_0^2}(c + x_0) \Leftrightarrow c = x_0^2 y_0 - x_0$

$f(f^{-1}(x)) = x$ $\quad \Rightarrow y(x) = \frac{1}{x_0^2}(x_0^2 y_0 - x_0 + x)$

2.3) $\begin{cases} y'' + y' - 2y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases}$

karakteristieke veelterm $\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = -2 \quad \lambda_2 = 1$

$\Rightarrow \varphi_1(x) = e^{-2x} \quad \varphi_2(x) = e^{x} \quad \varphi(x) = c_1 e^{-2x} + c_2 e^{x}$

$\left.\begin{array}{l} y(0) = 1 \Rightarrow c_1 + c_2 = 0 \\ y'(0) = 0 \Rightarrow -2c_1 + c_2 = 0 \end{array}\right\} \Rightarrow \begin{cases} c_1 = \frac{1}{3} \\ c_2 = \frac{2}{3} \end{cases} \quad y(x) = \frac{1}{3} e^{-2x}$

3) $y'' + \omega^2 y = 0 \quad \omega \in \mathbb{R}$ $\Rightarrow \lambda^2 + \omega^2 = 0 \Rightarrow \lambda_1 = i\omega,\ \lambda_2 = -i\omega$

opl van de $A\sin(\omega x + \alpha) \quad A, \alpha \in \mathbb{R}$ $\Rightarrow c_1 e^{i\omega x} + c_2 e^{-i\omega x}$

$x^2 + \omega^2 = (x - i\omega)(x + i\omega)$

algemene oplossing : $c_1 \sin\omega x + c_2 \cos\omega x \quad c_1, c_2 \in \mathbb{R}$

(niet homogeen)

4.2) $y'' + y = 3 + 6e^{-2\lambda}$

a) los homogene op
b) los niet-homogene op: $y'' + y = 3$ op
c) " $y'' + y = 6e^{-e\lambda}$

karakteristieke veelterm $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

(1) $\Rightarrow \varphi_1(x) = \sin x \quad \varphi_2(x) = \cos x$

particuliere oplossing : i) $\varphi_1(x) = e^{-2x}C$

$\leadsto \varphi_1'' + \varphi_1 = 6e^{-2x}$ ge weet dat de afgeleide dit ook gaat moeten bevatten

$\underbrace{4Ce^{-2x}}$ $+ Ce^{-2x} = 6e^{-2x}$

2e afgeleide van $\varphi_1(x) = e^{-2x}C$

$\Rightarrow C = \frac{6}{5} \Rightarrow \varphi_1(x) = \frac{6}{5} e^{-2x}$

ii) $\varphi_2 = C' \quad \varphi_2'' - \varphi_2 = 3 \Rightarrow \varphi_2 = 3$

$\Rightarrow \varphi(x) = c_1 \sin x + c_2 \cos x + \frac{6}{5} e^{-2x} + 3$

④ 4,8 ⑤ 1, 3, 4, 6, 11, 14 14-12-16

4.4) $y'' - 2y' + y = (x+1)e^{2x} + e^{x} - 1$

ⓐ Karakteristieke veelterm

$x^2 - 2x + 1 = 0$

~~$x^2 - 2x + 1 = 0$~~ $\quad x = 1$ (2)

$= (x-1)^2$

$\Rightarrow y = c_1 e^{x} + c_2 x e^{x}$, $c_1 c_2 \in \mathbb{R}$

ⓑ Particuliere oplossing van de niet-homogene

Afzonderlijk:

1. $y'' - 2y' + y = (x+1)e^{2x}$

$y(x) = (\alpha x + \beta)e^{2x}$

$e^{2x}\left(4(\alpha x+\beta) + 4\alpha - 4(\alpha x + \beta) - 2\alpha + (\alpha x + \beta)\right) = e^{2x}(x+1)$

$\Rightarrow \begin{cases} 4\alpha - 4\alpha + \alpha = 1 \\ 4\beta + 4\alpha - 4\beta - 2\alpha + \beta = 1 \end{cases} \Leftrightarrow \begin{cases} \alpha = 1 \\ \beta = -1 \end{cases}$

$\Rightarrow y = (x-1)e^{2x}$

2. $y = e^{x}\frac{x^2}{2}$

3. $y(x) = -1$

ⓒ Alg oplossing van de niet homogene oplossing

$y(x) = c_1 e^{x} + c_2 x e^{x} + (x-1)e^{2x} + e^{x}\frac{x^2}{2} - 1$

4.8) $\begin{cases} y'' - 2y' = e^{2x} + x^2 - 1 \\ y(0) = \frac{1}{8},\ y'(0) = 1 \end{cases}$

ⓐ $x^2 - 2x = 0$

$= (x)(x-2)$

$\varphi_1(x) = e^{0x} = 1$, $\varphi_2(x) = e^{2x}$

ⓑ particuliere oplossing van de niet-homogene veelterm

$y'' - 2y' = e^{2x} \overset{(12,4,3(2))}{\Rightarrow} \varphi_1(x) = xe^{2x}\cdot C$

$2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x} - 2(Ce^{2x} + 2Cxe^{2x}) = e^{2x}$

$\Rightarrow C = \frac{1}{2} \qquad \varphi_1(x) = \frac{x}{2}e^{2x}$

• $y'' - 2y' = x^2 - 1 = P(x)$

$\Rightarrow \varphi_2(x) = xe^{0x} \cdot C(x)$ met $C(x)$ zelfde graad als $P(x)$
(Stelling 12.4.3(2))

$$\varphi_2(x) = -\frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x$$

© algemene opl van de niet-homogene

$$y(x) = C_1 + C_2 e^{2x} + \frac{x}{2}e^{2x} - \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x$$

Theorie Bepaling particuliere oplossing van $y'' + py' + qy = R(x)$ (*)

① Verlaging van de orde (moet niet perse constante coëfficiënten)
- Bepaal de particuliere oplossing van de homogene vgl
- Stel $y = \varphi z$ (z is een nieuwe onbekende)

$$\varphi(x) z'' + (2\varphi'(x) + p\varphi(x)) z' = \underbrace{\varphi''(x) + \varphi'(x)p + \varphi(x)q}_{= 0 \text{ (door homogene oplossing)}}$$

$\varphi(x) \neq 0 \ \forall x$:

$$\Leftrightarrow z'' = \left| \frac{2\varphi'(x) + p\varphi(x)}{\varphi(x)} \right| z' = \frac{R(x)}{\varphi(x)} \quad (**)$$

- stel $u = z'$ (u = nieuwe onbekende)

$$u' + \left(\frac{2\varphi'(x) + p\varphi(x)}{\varphi(x)}\right) u = \frac{R(x)}{\varphi(x)}$$

$\Rightarrow$ 1ste orde in u

- Bepaal oplossing ψ van de 1e orde vgl

$\Rightarrow z' = \psi \Leftrightarrow z = \int \psi$ opl van (**)
$\Rightarrow y\int\psi$ oplossing van (*)

② Operatormethode $\Rightarrow$ met constante coëfficiënten
Stel : $x^2 + px + q = 0$ 2 verschillende wortels

$\lambda_1 = a + ib \quad \lambda_2 = a - ib$

$$\Rightarrow \varphi(x) = \frac{\sin bx\, e^{ax}}{b} \int R(t)\, e^{-at} \cos bt\, dt - \frac{\cos bx\, e^{ax}}{b} \int R(t) e^{-at} \sin bt\, dt$$

Stelling : Zij φ_j part. opl van $y' - \lambda_j y = R(x)$
$j = 1,2$

$$\Rightarrow \varphi = \frac{\varphi_2 - \varphi_1}{\lambda_2 - \lambda_1} \text{ part opl van (*)}$$

① $\lambda_1 \neq \lambda_2$ reëel : $\varphi(x) = \frac{1}{\lambda_2 - \lambda_1}\left(e^{\lambda_2 x}\int R(t) e^{-\lambda_2 t}\, dt - e^{-\lambda_1 x}\int R(t)\, e^{-\lambda_1 t}\, dt\right)$

5.1) $y'' + y = \tan x \qquad (-\frac{\pi}{2} < x < \frac{\pi}{2})$

Methode: verlaging van orde

a) algemene oplossing van de homogene vgl

~~$x^2+2x^2=0$~~ $\quad x^2 + 1 = (x-i)(x+i)$

$\Rightarrow \quad c_1 \cos x + c_2 \sin x \qquad c_1, c_2 \in \mathbb{R}$

1) $y = (\cos x)\, z$

$\cos x\, z'' - (2\sin x)\, z' = \operatorname{tg} x$

$z' = u \Rightarrow u' = -\frac{2\sin x}{\cos x} \cdot u = \frac{\sin x}{\cos^2 x}$

$u = e^{2\int \frac{\sin x}{\cos x}} \int \frac{\sin x}{\cos^2 x}\, e^{-2\int \frac{\sin x}{\cos x}}$

$\int \frac{\sin x}{\cos x} = -\ln \cos x$

$u = \frac{1}{\cos^2 x} \int \sin x = -\frac{1}{\cos x}$

$z = -\int \frac{1}{\cos x} = -\ln|\operatorname{tg}(\frac{x}{2} + \frac{\pi}{4})|$

$y = -\cos x \ln|\operatorname{tg}(\frac{x}{2} + \frac{\pi}{4})|$

Methode: operatormethode

$y = \sin x \int \frac{\sin t}{\cos t} \cos t\, dt - \cos x \int \frac{\sin^2 t}{\cos t}\, dt$

$= -\cos x \int \frac{1}{\cos t}\, dt = -\cos x \ln|\operatorname{tg}(\frac{x}{2} + \frac{\pi}{4})|$

6.3) $y'' - 2y' + y = \frac{e^x}{(x-1)^2} \qquad (x > 1)$

$x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 \Rightarrow \lambda = 1$ = dubbele wortel

$\Rightarrow \varphi_1 = e^x$ en $\varphi_2 = xe^x$

1) $y = e^x z \Rightarrow$
$$y' = e^x z' + e^x z$$
$$y'' = e^x z'' + e^x z' + e^x z' + e^x z$$
$$= e^x z'' + 2e^x z' + e^x z$$

$e^x z'' + \cancel{2e^x z'} + \cancel{e^x z} - \cancel{2e^x z'} - \cancel{2e^x z} + \cancel{e^x z} = \frac{e^x}{(x-1)^2}$

$\Rightarrow z'' = \frac{1}{(x-1)^2} \Rightarrow z' = \frac{-1}{x-1} \Rightarrow z = -\ln(x-1)$

$\Rightarrow y = -e^x \ln|x-1|$

Algemene oplossing van de niet-homogene:

$\varphi = C_1 e^x + C_2 x e^x - e^x \ln|x-1|$

15-12-16

7.1) $xy'' - (1+x)y' + y = 0$

a) $\Rightarrow$ $y = e^x$ is een oplossing (som van de coëfficiënten is 0)

b) $y'' - \left(\frac{1+x}{x}\right)y' + \frac{y}{x} = 0$

$y = e^x z \Rightarrow e^x z'' + \left(2e^x - \left(\frac{1+x}{x}\right)e^x\right)z' = 0$

$z' = u \Rightarrow u' + \left(2 - \frac{1+x}{x}\right)u = 0$

$u' + \left(\frac{x-1}{x}\right)u = 0$

$u = e^{-\int \frac{x-1}{x}} = e^{-x+\ln x} = xe^{-x}$

$z = \int xe^{-x} = -xe^{-x} + \int e^{-x} = (-x-1)e^{-x}$

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$\Rightarrow y = -x-1$

algemene oplossing: $c_1 e^x + c_2(x+1) \qquad c_1, c_2 \in \mathbb{R}$

(oefeningen voor vandaag: 7 1,2,3 8 7 6,7)

7.2) $(1-x^2)y'' - 2xy' + 2y = 0$

a) $y = x$ is een oplossing (op zicht)

b) $y'' - \frac{2xy'}{(1-x^2)} + \frac{2y}{(1+x^2)} = 0$

$y = xz \Rightarrow xz'' + \left(2 - \frac{2x^2}{1-x^2}\right)z' = 0$

$z' = u \Rightarrow u' + \left(\frac{2}{x} - \frac{2x}{1-x^2}\right)u = 0$

$u = e^{\left(-\int \frac{2}{x} - \frac{2x}{1-x^2}\right)} = e^{-2\ln x - \ln(x^2-1)}$

$= \frac{1}{(x^2-1)x^2}$

$z = \int \frac{1}{(x^2-1)x^2} = -\int \frac{1-x^2}{(x^2-1)x^2} + \int \frac{x^2}{(x^2-1)x^2}$

$= -\int \frac{1}{x^2} + \int \frac{1}{x^2-1}$

$= \frac{1}{x} - \operatorname{argtanh} x$

$y = 1 - x \operatorname{argtanh} x$

$\Rightarrow$ algemene oplossing: $y = c_1 x + c_2(1 - x\operatorname{argtanh} x)$

$c_1, c_2 \in \mathbb{R}$

7.3) $y'' + 2\tan x y' - y = 0$

$y'' + 2\frac{\sin x y'}{\cos x} - y = 0 \;=\; y''\cos x + 2\sin x y' - y\cos x = 0$

a) $\Rightarrow y = \sin x$

b) I) $y = \sin x z \Rightarrow \sin x z'' + (2\cos x + 2\tan x \sin x) z' = 0$

II) $u = z'$ $\quad u' + \left(\frac{2\cos x}{\sin x} + 2\tan x\right) u = 0$

$u = e^{-2\int \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}}$ $\quad = -2\int \frac{1}{\sin x} d\sin x - \frac{1}{\cos x} d\cos x$

$= e^{-2\ln\left(\frac{\sin x}{\cos x}\right)}$

$= \frac{\cos^2 x}{\sin^2 x}$

III) $z = \int u = \int \frac{\cos^2 x}{\sin^2 x} = \int \frac{1}{\sin^2 x} - \int 1 = \frac{-\cos x}{\sin x} - x$

IV) $y = -\cos x - x\sin x$

algemene oplossing: $c_1 \sin x + c_2(-\cos x - x\sin x) \quad c_1, c_2 \in \mathbb{R}$

8) $y'' - \left(\frac{2x+3}{x+1}\right) y' + \left(\frac{x+2}{x+1}\right) y = (x+1)e^{2x}$

a) algemene oplossing van de homogene vgl

a) $y = e^x$ (som van coeff $= 0$)
b) $y = e^x z$

$\cancel{e^x} z'' + \left(2\cancel{e^x} - \frac{2x+3}{x+1}\cancel{e^x}\right) z' = 0$

$z' = u$

$\Rightarrow u' - \frac{u}{x+1} = 0$

$u = e^{\int \frac{1}{x+1}} = x+1$

$z = \frac{x^2}{2} + x \quad , \quad y = e^x\left(\frac{x^2}{2} + x\right)$

$\Rightarrow y = c_1 e^x + c_2 e^x\left(\frac{x^2}{2} + x\right)$

b) particuliere oplossing van de niet-homogene

$y = e^x z$

$\cancel{e^x} z'' + \left(2\cancel{e^x} - \frac{2x+3}{x+1}\cancel{e^x}\right) z' = (x+1)e^{\cancel{2}x}$

$\rightarrow u = z'$

$u' - \frac{u}{x+1} = (x+1)e^x$

$u = e^{\int \frac{1}{x+1}} \cdot \int (x+1)e^x \cdot e^{-\int \frac{1}{x+1}}$

$= (x+1)\int \frac{(x+1)e^x}{(x+1)} = (x+1)e^x$

$z = \int u = xe^x \Rightarrow y = xe^{2x}$

c) oplossing: van de niet-homogene

$c_1 e^x + c_2 e^x\left(\frac{x^2}{2} + x\right) + xe^{2x}$

Oef 3 blz 134 17/11/16

1 $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)} = \lim_{N\to\infty} \sum_{N=1}^{N} \frac{1}{(3n-2)(3n+1)}$

$\boxed{\text{Splitsen in partielbreuken } = \frac{1}{3}\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)}$

$= \frac{1}{3} \lim_{N\to\infty} \sum \frac{1}{3n-2} - \sum_{n=1}^{N} \frac{1}{3n+1}$

$= \frac{1}{3} \lim_{n=0}^{N-1} \frac{1}{3(n+1)-2} - \lim_{n=1}^{N} \frac{1}{3n+1}$

$= \frac{1}{3} \lim_{n\to\infty} \left(1 - \frac{1}{3N+1}\right) = \frac{1}{3}$

3 $\sum_{n=3}^{\infty} \frac{4n-3}{n^3-4n} = \frac{1}{8} \lim_{N\to\infty} \left(\sum_{n=3}^{N} \frac{6}{n} + \sum_{n=3}^{N} \frac{5}{n-2} - \sum_{n=3}^{N} \frac{11}{n+2} \right)$

$= \frac{1}{8} \lim_{N\to\infty} \left(6 \sum_{n=3}^{N} \frac{1}{n} + 5 \sum_{n=3}^{N} \frac{1}{n-2} - 11 \sum_{n=3}^{N} \frac{1}{n+2} \right)$

$= \frac{1}{8} \lim_{N\to\infty} \left(6 \sum_{n=3}^{N} \frac{1}{n} + 5 \left(\sum_{n=3}^{N} \frac{1}{n} + 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N-1} \right) \right.$

$\left. - 11 \left(\sum_{n=3}^{N} \frac{1}{n} - \frac{1}{3} - \frac{1}{4} + \frac{1}{N+1} + \frac{1}{N+2} \right) \right)$

$= \frac{1}{8} \left(5 + \frac{5}{2} + \frac{11}{3} + \frac{11}{4} \right) = \frac{167}{96}$

4 $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ wetende dat $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$

$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

$\boxed{\text{oneven getallen = alle getallen - even getallen}}$

$= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$

$= \frac{\pi^6}{6} - \frac{\pi^6}{24}$

$= \frac{\pi^2}{8}$

5 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{k=1}^{\infty} \frac{(-1)^{2k}}{(2k)^2} + \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)^2}$

$\boxed{-1 \text{ en } +1 \text{ splitsen door deze methode}}$

$= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$

$= \frac{\pi^2}{24} - \frac{\pi^2}{8} = \frac{-\pi}{12}$

Reeksen zonder negatieve termen

1) Majorantenregel

$\sum x_n \ll \sum y_n$ $\quad \sum y_n$ conv $\Rightarrow \sum x_n$ conv

2) Meetkundige reeks

$\sum_{n=0}^{\infty} R^n = \ldots$ $\quad$ convergeert voor $r < 1$

3) Hyperharmonische

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad$ convergeert als $p > 1$
divergeert als $p \leq 1$

4) d'Alembert

$$\begin{cases} \lim \frac{x_{n+1}}{x_n} < 1 \Rightarrow \text{convergent} \\ \qquad > 1 \Rightarrow \text{divergent} \end{cases}$$

Cauchy

$$\begin{cases} \lim \sqrt[n]{x_n} < 1 \Rightarrow \text{convergent} \\ \qquad > 1 \Rightarrow \text{divergent} \end{cases}$$

Raabe

$$\begin{cases} \lim n\left(\frac{x_n}{x_{n+1}} - 1\right) > 1 \Rightarrow \text{convergent} \\ \qquad < 1 \Rightarrow \text{divergent} \end{cases}$$

5) Integraaltest

$f : [1, +\infty[\to [0, +\infty[$

$\sum_{n \geq 1} f(n)$ convergent $\Rightarrow \int f(n)\,dx$ convergent

Oef 4 blz 131

1) $\sum_{n\geq 1} \frac{1}{n^n}$ $\quad \frac{1}{n^n} \leq \frac{1}{2^n} \Rightarrow \sum \frac{1}{n^n} \ll \sum \frac{1}{2^n}$
$\Rightarrow$ convergent $\quad$ "meetkundige reeks met $R = \frac{1}{2}$

3 $\sum_{n\geq 1} n \tan \frac{\pi}{2^n+1}$ $\quad \boxed{\lim_{x\to 0} \frac{\tan x}{x} \overset{\text{l'H}}{=} \frac{1}{\cos^2 x} = 1}$

d'Alembert: $\lim_{n\to\infty} \dfrac{(n+1)\dfrac{\tan(\pi/2^n+2)}{\pi/2^n+2} \cdot \dfrac{\pi}{2^n+2}}{n\dfrac{\tan(\pi/2^n+1)}{\pi/2^n+1} \cdot \dfrac{\pi}{2^n+1}}$ ...

5 $\sum_{n\geq 1} \frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}$ $\quad$ cauchy: $\sqrt[n]{\frac{\left(\frac{n+1}{n}\right)^{2n}}{3^n}} = \frac{\left(\frac{n+1}{n}\right)^2}{3} < 1$
$\Rightarrow$ convergent

7 $\sum_{n\geq 1} \frac{1}{\sqrt{n^2+2n}}$

quotiëntregel: $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2+2n}}} = \lim_{n\to\infty} \sqrt{1+\frac{2}{n}} = 1$

convergentiegedrag $\sum_{n\geq 1} \frac{1}{\sqrt{n^2+2n}}$ en $\sum_{n\geq 1} \frac{1}{n}$ zijn gelijk

$\to$ divergent

9 $\sum_{n\geq 1} \frac{1}{(n+1)\ln^2(n+1)}$

Integraaltest: $f(x) = \frac{1}{(x+1)\ln^2(x+1)}$

$$I_n = \int_1^n \frac{1}{(x+1)\ln(x+1)}\,dx$$
$$= \int_1^n \frac{d(\ln(x+1))}{\ln^2(x+1)}$$
$$= \left[-\frac{1}{\ln(x+1)}\right]_1^n$$
$$= \frac{1}{\ln 2} - \frac{1}{\ln(n+1)}$$
$$= \frac{1}{\ln 2}$$ => de reeks is dus convergent

10 $\sum_{n\geq 0} \frac{1\cdot 4\cdot 7\cdot \ldots \cdot(3n+1)}{(n+1)!}x^n$

D'Alembert: $$\lim_{n\to\infty} \frac{\frac{\boxed{1\cdot 4\cdot 7\ldots(3n+4)}}{\boxed{(n+2)!}}\,x^{\boxed{n+1}}}{\frac{\boxed{1\cdot 4\cdot 7\ldots(3n+1)}}{\boxed{(n+1)!}}\,\boxed{x^n}} = \lim_{n\to\infty}\frac{3n+4}{3n+1}x = 3x$$?

$\cancel{\frac{3n+1+3}{3n+1}}$ $\frac{3n+1+3}{3n+1}$ $\Rightarrow 1 + \frac{3}{3n+1}$

→ divergent voor $x > 1/3$
→ convergent voor $x < 1/3$

Wat als $x = 1/3$? : Raabe:

$$\lim_{n\to\infty} n\left(\frac{n+2}{(3n+4)1/3} - 1\right) = \lim_{n\to\infty}\frac{2n}{3n+4} = \frac{2}{3} < 1 \to \text{divergent}$$

$\sum z_n$ • Als $\sum|z_n|$ convergeert → absoluut convergent
• anders: betrekkelijk
Leibniz: (p_n) dalende rij positieve getallen met $p_n \to 0$
=> $\sum(-1)^n p_n$ is convergent

Oef 5 blz 131

1 $\sum_{n\geq 1}\frac{(-1)^{n+1}}{\ln(n+1)}$ absoluut: $\sum_{n\geq 1}\left|\frac{(-1)^{n+1}}{\ln(n+1)}\right|$

$\boxed{= \sum\frac{1}{\ln(n+1)} \gg \sum\frac{1}{n}}$
=> divergent

betrekkelijk: $p_n = \frac{1}{\ln(n+1)} \searrow 0$ => betrekkelijk convergent

3 $\sum_{n\geq 1}\frac{\sqrt[n]{2}}{n}x^n$ $x\in\mathbb{R}$

d'Alembert voor abs reeks: $$\lim_{n\to\infty}\frac{\frac{2^{\frac{1}{n+1}}|x|^{n+1}}{n+1}}{\frac{2^{\frac{1}{n}}|x|^n}{n}}$$
$$= \lim_{n\to\infty} 2^{\frac{1}{n+1}-\frac{1}{n}}\cdot\frac{|x|n}{n+1}$$
$$= \lim_{n\to\infty} 2^{\frac{n-n-1}{n(n+1)}}\cdot\frac{|x|\,n}{n+1} = |x|$$

Abs voor conv voor $|x|<1$
divergent voor $|x|>1$

$\boxed{|x| = 1 \text{ ? checken voor } -1 \text{ en } +1}$

4 $\sum_{n\geq 1} \frac{x^{n^2}}{n!}$ $(x \in \mathbb{R})$

absoluut: $\sum \frac{(|x|)^n}{n!}$

d'Alembert:

$\boxed{\frac{n!}{(n+1)!} = \frac{1}{n+1}}$

$$\lim_{n\to\infty} \frac{|x|^{(n+1)^2} n!}{(n+1)!\,|x|^{n^2}}$$

$$= \lim \frac{|x|^{n^2+2n+1-n^2}}{n+1}$$

$$= \lim \frac{|x|^{2n+1}}{n+1} = \begin{cases} 0 & |x| \leq 1 \\ \infty & |x| > 1 \end{cases}$$

=> absoluut convergent voor $|x| \leq 1$

niet abs voor $|x| > 1$: $\frac{x^{n^2}}{n!} \not\to 0 \to$ divergent

8 $\sum \sin(n\pi + \frac{1}{n}) = \sum (-1)^n \sin\frac{1}{n}$

$$\boxed{\begin{aligned} \sin(n\pi + \tfrac{1}{n}) &= \underbrace{\sin(n\pi)}_{\to 0}\cos(\tfrac{1}{n}) + \sin(\tfrac{1}{n})\underbrace{\cos(n\pi)}_{\to 1/-1} \\ &= (-1)^n \sin(\tfrac{1}{n}) \end{aligned}}$$

Betrekkelijk: Leibniz

absoluut convergent?

vergelijken met $\sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim \frac{\cos\frac{1}{n} \cdot -\frac{1}{n^2}}{-1/n}$$

$$= \lim \cos\frac{1}{n} = 1$$

$\sum \frac{1}{n}$ en $\sum \sin n$: zelfde convergentiegedrag : divergent

6 $\sum_{n\geq 1} (-1)^n \frac{x(x+1)\ldots(x+n-1)}{n!}$ $(x>0)$

$x \geq 1$: $\left| \frac{(-1)^n x(x+1)\ldots(x+n-1)}{n!} \right|$

$$= \prod_{j=1}^{n} \frac{(x-1)+j}{j} \geq 1$$

$\not\to$ dus niet convergent

$x < 1$: absoluut $\lim_{n\to\infty} n\left(\frac{n+1}{n+x} - 1\right)$

$$= \lim_{n\to\infty} n\left(\frac{1-x}{n+x}\right) = 1-x < 1$$

=> niet absoluut convergent

betrekkelijk:

Leibniz met $p_n = \ldots$

$$= \prod_{j=1}^{n} \frac{j+(1-x)}{j}$$

$\to$ dalend

Oef 2 blz 149

Taylorreeksen

1 Herleiden tot meetkundige reeks

$$\frac{1}{x^2+3} = \frac{1}{3}\cdot\frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2} = \frac{1}{3}\cdot\frac{1}{1-\left(-\left(\frac{\sqrt{x}}{3}\right)^2\right)}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{\sqrt{3}}\right)^{2n} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}x^{2n}$$

$R = \sqrt{3}$

geen convergentie in de eindpunten want meetkundige reeks herleiden tot meetkundige reeks

$$R = \frac{1}{\lim\limits_{n\to+\infty}\sqrt{\frac{1^{n/2}}{\sqrt{3}}}} = \lim_{n\to\infty}\frac{(1)^{n/2}/\sqrt{3}}{(1)^{\frac{2+n}{2}}/\sqrt{3^3}} = \frac{(1)^{-1}}{\sqrt{3}} = \sqrt{3}$$

2

$$\frac{a+x}{b+x} = (a+x)\cdot\frac{1}{(b+x)} = \frac{(a+x)}{b}\cdot\frac{1}{1+\frac{x}{b}}$$

$$= \frac{a+x}{b}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{b^n}$$

$$= \frac{a}{b}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{b^n} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{b^{n+1}}$$

$$= \text{''} \ldots - \frac{(-1)^{n+1}\ldots}{\ldots}$$

$$= \ldots - \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{b^n}$$

$$= \frac{a}{b} + \left(\frac{a}{b}-1\right)\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{b^n}$$

geen convergentie in de eindpunten

3

$$\frac{1}{1+x+x^2+x^3+x^4}$$

tip: gebruik $1+x+\ldots+x^n = \frac{1-x^{n+1}}{1-x}$

$$= \frac{1-x}{1-x^{(n+1)}} = (1-x)\cdot\frac{1}{1-x^{n+1}} = (1-x)\sum_{n=0}^{\infty}\left(x^{n+1}\right)^n$$

$$= (1-x)\sum_{n=0}^{\infty}(x^5)^n = \sum x^{5n} - \sum x^{5n+1}$$

Schrijven als 1 reeks:

$\sum_{n=0}^{\infty} c_n x^n$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad x \in \mathbb{R}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad x \in \mathbb{R}$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad \alpha \in \mathbb{R} \text{ en } (-1 < x < 1)$$

$$= \binom{\alpha}{n} = \frac{1}{n!} \prod_{j=0}^{n-1} (\alpha - j)$$

Regels

$\sum_{n=0}^{\infty} a_n x^n \qquad a_n \in \mathbb{C}, x \in \mathbb{R}$

convergentiestraal $R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{a_n}} \in [0, 8]$

als deze bestaat $\rightarrow = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

- absolute convergentie voor $|x| < R$
  als $R = \infty \Rightarrow \forall x \in \mathbb{R}$
- geen convergentie voor $|x| > R$
- convergentie in eindpunten $x = \pm R$ $(R < \infty)$ worden apart behandeld.

$$\ln(x+1) = \int_0^x \frac{1}{1+t}\, dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n \, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)\, x^{n+1}}{n+1}$$

$$\cos x = \sin x \quad = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1} \qquad x \in \mathbb{R}$$

vervolg 3.2

$$c_n \begin{cases} 1 & n = 5k \quad k \in \mathbb{N} \\ -1 & n = 5k+1 \quad k \in \mathbb{N} \\ 0 & \text{anders} \end{cases} \Rightarrow R = 1$$

Als de macht van de vorm 5k is, is de coëff van de eerste term = 1, als 5k+1 is de coëfficiënt van de 2e term = -1. Andere machten komen niet voor

$$\frac{1}{(1+x^2)} = -\left(\frac{1}{1+x}\right)' = -(1 - x + x^2 - x^3 + \ldots)^{3'}$$

$$= 1 - 2x + 3x^2 - \ldots$$

5 $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$\rightarrow$ reeksontwikkeling sinus.

29-11-16

Hoofdstuk 10

3) $F(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \quad (-1 < k < 1)$

$\rightarrow$ nemen x als vaste constante

$$\int_0^{\pi/2} \frac{1\,dx}{\sqrt{1-k^2\sin^2 x}} = \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2\sin^2 x \; \frac{1\cdot 3}{2\cdot 4}k^4\sin^4 x + \dots \right) dx$$

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \left[\frac{-\sin^{2n-1}x\cos x}{2n}\right]_0^{\pi/2} + \frac{2n-1}{2n}\int_0^{\pi/2} \sin^{2n-2} x\,dx$$
$$= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{\pi}{2}$$

$$\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\pi}{2}\left(1 + \sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 k^{2n}\right)$$

~~Theorie~~

4) a) $\sqrt{\frac{1+x}{1-x}}$ (tot en met x^2)

$= (1+x)^{-1/2}\cdot(1-x)^{-1/2}$

$= \left(1 + \frac{x}{2} - \frac{x^2}{8} + O(x^3)\right)\left(1 + \frac{x^2}{2} + \frac{3x^3}{8} + O(x^3)\right)$

$= 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{x}{2} + \frac{x^2}{4} - \frac{x^2}{8} + O(x^3)$

$= 1 + x + \frac{x^2}{2} + O(x^3)$

b) $\cosh x \sin x$ (tot en met x^3)

$= \left(1 + \frac{x^2}{2} + O(x^4)\right)\left(x - \frac{x^3}{6} + O(x^5)\right)$

$= x - \frac{x^3}{6} + \frac{x^3}{2} + O(x^4) = x + \frac{x^3}{3} + O(x^4)$

c) $\tan x$ tot en met x^5

$= \frac{\sin x}{\cos x}$

$= \cancel{\dfrac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - O(x^6)}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - O(x^5)}}$

$= a_0 + a_1 x + a_2 x^2 + \dots$

$\Rightarrow (a_0 + a_1 x + a_2 x^2 + \dots)\cdot\cos x = \sin x$

$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \right) \quad \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)$

$\Rightarrow$ Stelsel maken

$a_0 = 0$

$a_1 = 1$

$a_2 = -\frac{1}{2}a_0 = 0$

$-\frac{1}{6} = a_3 - \frac{a_1}{2}$

$\frac{a_0}{4!} - \frac{a_2}{2} + a_4 = 0$

$\frac{1}{5!} = a_5 - \frac{a_3}{2} + \frac{a_1}{4!}$

$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^6)$

d) $\dfrac{e^x + \sin x - 1}{\ln(1+x)}$ (tem x^3)

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + O(x^4)$

$\Rightarrow (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + O(x^4)) \cdot \left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + O(x^5)\right)$

$= \left(2x + \dfrac{x^2}{2} + \dfrac{x^4}{24} + O(x^5)\right)$

Reeksontwikkeling van $e^x + \sin x - 1$, linkerlid uitwerken en stelsel gelijkstellen aan LL

$a_0 = 2$

$-\dfrac{a_0}{2} + a_1 = \dfrac{1}{2}$

$\dfrac{a_0}{4} + \dfrac{a_1}{3} - \dfrac{a_2}{2} + a_3 = \dfrac{1}{24}$

$a_0 = 2$

$a_1 = 3/2$

$a_2 = 1/12$

$a_3 = 1/12$