Analytische Getaltheorie
Examen – 19.1.2019

(Veel succes!)

Instructions: Use a separate paper sheet to answer each question. Make sure you write your name on each of the sheets.

Problem 1

Without using the PNT (you may only use results at the level of Chebyshev and/or Mertens estimates), determine whether the series $\sum_{n\geq 2} \frac{\Lambda(n)}{n\log n}$ converges, and obtain an asymptotic obtain estimate for

$$S(x) = \sum_{1<n\leq x} \frac{\Lambda(n)}{n\log n}$$

with as good an error term you can get without appealing to the PNT.

Problem 2

Let $F(s) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n/3)}{n^s}$.

(a) Express $F(s)$ in terms of the Riemann zeta function and/or Dirichlet L-series.

(b) Determine, with proof, the abscissas of absolute convergence σ_a and convergence σ_c of $F(s)$.

(c) Show that $F(s)$ has a meromorphic continuation to the half-plane $\sigma > 0$ and determine (if any) all its poles in this half-plane.

Problem 3

Let $k(n)$ be the squarefree kernel of the integer n, that is, $k(n) := \prod_{p|n} p$.

(a) Compute the abscissa of convergence of the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{k(n)n^s}$.

(b) Deduce the following bound on $N(x,y) := \text{card}\{n \leq x : k(n) \leq y\}$. For each $\varepsilon > 0$, we have uniformly for $1 \leq y \leq x$,

$$N(x,y) \ll_\varepsilon yx^\varepsilon.$$

Problem 4

The goal of this exercise is to find a good asymptotic formula for the summatory function of $3^{\omega(n)}$.

(a) Find an asymptotic formula for the summatory function of $d_3(n) = 1*1*1(n)$, that is, estimate $\sum_{abc\leq x} 1$. Try to get an error term as low as possible.

(b) Define g through $3^{\omega(n)} = 1*1*1*g$. Calculate g at prime powers and show that for any $\varepsilon > 0$, we have $|g(n)| \ll_\varepsilon n^\varepsilon$.

(c) Use parts (a) and (b) to find an asymptotic formula (with an error term as low as possible) for

$$\sum_{n\leq x} 3^{\omega(n)}.$$

In deducing this, you may use the following result on squarefull integers (which we showed in the exercises). Let $S = \{n \in \mathbb{N} : p \mid n \Rightarrow p^2 \mid n\}$. Then, $S(x) := \sum_{\substack{n\leq x \\ n\in S}} 1 = O(x^{1/2})$.