# Wiskundige Logica 2

Master Wiskunde - UGent
Tweede examenperiode 2018-2019

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## Closed book questions

**Exercise 1.** (1.5 points each)

1. Prove that $L_\alpha$ is transitive for every $\alpha$.
2. Recall that $\mathrm{rk}(x)$ is the least $\alpha$ such that $x \in V_{\alpha+1}$. Prove that $\mathrm{rk}(x) = \sup\{\mathrm{rk}(y)+1 : y \in x\}$.
3. Prove that if $x$ is an infinite set and $s$ is the set of finite subsets of $x$ then $\mathrm{card}(x) = \mathrm{card}(s)$.

**Exercise 2.** Let $\omega := \bigcap\{x : 0 \in x \wedge \forall n \in x\ (n+1) \in x\}$. Show (1 point each):

1. $\omega \in V$,
2. $\omega \subseteq \mathsf{Ord}$,
3. $(\omega, 0, +1) \models \forall x \subseteq \omega\ \Big((0 \in x \wedge \forall n \in x\ (n+1 \in x)) \to (x = \omega)\Big)$,
4. $\omega \in \mathsf{Ord}$.

**Exercise 3.** Prove the following theorem (1.5 points):

Let $G(w, \vec{y})$ be a definite term, and let $F(x, \vec{y})$ be the canonical term defined by $\in$-recursion with $G$:

$$\forall x\ F(x, \vec{y}) = G(\{(z, F(z, \vec{y})) \mid z \in x\}, \vec{y}).$$

Then the term $F(x, \vec{y})$ is definite.

# Open book questions

**Exercise 1.** (2.5 points)
Show that there is, up to equality of classes (see Definition 3 of the coursenotes), a unique class $A$ with the following property:

$$(\forall x) \qquad x \in A \Leftrightarrow x \subseteq A \wedge x \text{ transitive.}$$

**Exercise 2.** (2.5 points)
Consider the following three statements

$$\psi_1 \equiv (\forall \kappa, \lambda \in \text{Card})\, \lambda < \kappa \Rightarrow \kappa^\lambda \leq \lambda^\kappa.$$
$$\psi_2 \equiv (\forall \kappa, \lambda \in \text{Card})\, \kappa^\lambda = \lambda^\kappa.$$
$$\psi_3 \equiv (\forall \kappa, \lambda \in \text{Card})\, \lambda < \kappa \Rightarrow \kappa^\lambda < \lambda^\kappa.$$

For each of these statements, give either a ZFC-proof of the statement or a ZFC-proof of the negation of the statement.

**Exercise 3.** (2.5 points)
Let $\kappa \in \text{Card}$ be regular and uncountable and let $g : \kappa \to \kappa$ be increasing and unbounded.
Show, for every function

$$f : \kappa \to \{A \subseteq \kappa : \text{card}(A) < \kappa\},$$

that

$$\{\alpha < \kappa : (\forall \beta < \alpha)\ g(\alpha) \notin f(\beta)\} \in \mathcal{C}_\kappa.$$

**Exercise 4.** (2.5 points)
Let $\kappa \in \text{Card}$ and let $(M_1, \in)$, $(M_2, \in)$ be two transitive ZFC-models with $\kappa \in M_1 \subseteq M_2$.
Suppose that

$$(\kappa \text{ is a singular cardinal})^{M_1}.$$

Prove that

$$(\kappa \text{ is a singular cardinal})^{M_2}.$$

Success!