<u>Hoofdstuk 6: Relatierekenen:</u>

<u>Oefening 1:</u>

$$
\begin{aligned}
d \notin \{x \mid 0 . e\} &\equiv \langle \text{def} \notin \rangle\ \neg(d \in \{x \mid 0 . e\} \\
&\equiv \langle \text{axV1} \rangle\ \neg(\exists(x \mid a . d = e\} \\
&\equiv \langle \text{ax16b} \rangle\ \neg 0 \\
&\equiv \langle \text{st7} \rangle\ 1
\end{aligned}
$$

<u>Oefening 2:</u>

$$
\begin{aligned}
x \in \{x \mid r\} &\equiv \langle \text{dummy hernoemen, met v niet vrij in e} \rangle\ x \in \{v \mid r[x := v]\} \\
&\equiv \langle \text{axV1} \rangle\ \exists(v \mid r[x := v] . v = x) \\
&\equiv \langle \text{st76} \rangle\ \exists(v . r[x := v] \wedge v = x) \\
&\equiv \langle \text{st76} \rangle\ \exists(v \mid v = x . r[x := v]) \\
&\equiv \langle \text{ax17b} \rangle\ r[x := v]\ [v := x] \\
&\equiv \langle \text{subst. + v niet vrij in r} \rangle\ 1
\end{aligned}
$$

<u>Oefening 3: a:</u>

$\{x : X \mid p \vee q\} = \{x : X \mid p\} \cup \{x : X \mid q\}$

$v \in \{x : X \mid p \vee q\} \equiv v \in \{x : X \mid p\} \cup \{x : X \mid q\}$

$$
\begin{aligned}
v \in \{x : X \mid p\} \cup \{x : X \mid q\} &\equiv \langle \text{axV5} \rangle\ v \in \{x : X \mid p\} \vee v \in \{x : X \mid q\} \\
&\equiv \langle \text{axV1} \rangle\ \exists(x : X \mid p . v = x) \vee \exists(x : X \mid q . v = x) \\
&\equiv \langle \text{ax19b} \rangle\ \exists(x : X \mid p \vee q . v = x) \\
&\equiv \langle \text{axV1} \rangle\ v \in \{x : X \mid p \vee q . v = x\}
\end{aligned}
$$

wegens veralgemening (metastelling)

$\forall(v . v \in \{x : X \mid p \vee q\} \equiv v \in \{x : X \mid p\} \cup \{x : X \mid q\}$

$\equiv \langle \text{axV2} \rangle\ \{x : X \mid p \vee q\} = \{x : X \mid p\} \cup \{x : X \mid q\}$

<u>Oefening 4: a:</u>

$X \cup Y = Y \cup X$

$$
\begin{aligned}
x \in X \cup Y &\equiv \langle \text{axV5} \rangle\ x \in X \vee x \in Y \\
&\equiv \langle \text{axV5} \rangle\ x \in Y \cup X
\end{aligned}
$$

wegens veralgemening

$\forall(x . x \in X \cup Y \equiv x \in Y \cup X)$

$\equiv \langle \text{axV2} \rangle\ X \cup Y = Y \cup X$

<u>Oefening 6: a:</u>

$co(X \cup Y) = coX \cap coY$

$$
\begin{aligned}
v \in co(X \cup Y) &\equiv \langle \text{axV4} \rangle\ v \in \Omega \wedge v \notin (X \cup Y) \\
&\equiv \langle \text{def} \notin \rangle\ v \in \Omega \wedge \neg(v \in (X \cup Y)) \\
&\equiv \langle \text{axV5} \rangle\ v \in \Omega \wedge \neg(v \in X \vee v \in Y) \\
&\equiv \langle \text{st22b} \rangle\ v \in \Omega \wedge (\neg(v \in X) \wedge \neg(v \in Y)) \\
&\equiv \langle \text{st23} \rangle\ (v \in \Omega \wedge \neg(v \in X)) \wedge (v \in \Omega \wedge \neg(v \in Y)) \\
&\equiv \langle \text{def} \notin \text{, axV4} \rangle\ v \in coX \wedge v \in coY \\
&\equiv \langle \text{axV6} \rangle\ v \in (coX \cap coY)
\end{aligned}
$$

wegens veralgemening +axV2

Oefening 10: a:

$X \setminus (Y \cup Z) = (X \setminus Y) \cup (X \setminus Z)$
$v \in X \setminus (Y \cup Z) \quad \equiv <\text{axV7}> v \in X \wedge v \notin (Y \cup Z)$
$\equiv <\text{def } \notin, \text{axV5}> v \in X \wedge \neg(v \in Y \vee v \in Z)$
$\equiv <\text{st23}> (v \in X \wedge \neg(v \in Y)) \wedge (v \in X \wedge \neg(v \in Z))$
$\equiv <\text{axV7, def } \notin> v \in (X \setminus Y) \wedge x \in (X \setminus Z)$
$\equiv <\text{axV7}> v \in ((X \setminus Y) \cup (X \setminus Z))$
wegens veralgemening + axV2

Oefening 11: a:

$(X \subseteq Y) \wedge (Y \subseteq X) \quad \equiv <\text{axV3}> \forall (x \mid x \in X . x \in Y) \wedge \forall (x \mid x \in Y . x \in X)$
$\equiv <\text{ax23}> \forall (x . x \in X => x \in Y) \wedge \forall (x . x \in Y => x \in X)$
$\equiv <\text{ax18a}> \forall (x . (x \in X => x \in Y) \wedge (x \in Y => x \in X))$
$\equiv <\text{st57}> \forall (x . x \in X \equiv x \in Y)$
$\equiv <\text{axV2}> X = Y$

Oefening 12:

$(x,y) \in (R \circ S) \quad \equiv \exists z . (x,z) \in R \wedge (z,y) \in S$
$R = \{(b,b),(b,c),(c,d)\}$
$S = \{(b,c),(c,d),(d,b)\}$
$R \circ S = \{(b,c),(b,d),(c,b)\}$
$R \circ R = \{(b,b),(b,c),(b,d)\}$

Oefening 13:

$(R \circ S)^{-1} = S\text{-}1 \circ R\text{-}1$
$(x,z) \in (R \circ S)^{-1} \quad \equiv <\text{def -1 (omgekeerde relatie)}> (z,x) \in R \circ S$
$\equiv <\text{def } \circ \text{ (samengestelde relatie)}> \exists (y . (z,y) \in R \wedge (y,z) \in S)$
$\equiv <\text{def -1}> \exists (y . (y,z) \in R\text{-}1 \wedge (x,y) \in S\text{-}1)$
$\equiv <\text{def } \circ> (x,z) \in (S\text{-}1 \circ R\text{-}1)$
wegens veralgemening + axV2

Oefening 14: a:

$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$
$(x,z) \in R \circ (S \cup T) \quad \equiv <\text{def } \circ> \exists (z . (x,z) \in R \wedge (z,y) \in (S \cup T))$
$\equiv <\text{axV5}> \exists (z . (x,z) \in R \wedge ((z,y) \in S \vee (z,y) \in T))$
$\equiv <\text{st28}> \exists (z . ((x,z) \in R \wedge (z,y) \in S) \vee ((x,z) \in R \wedge (z,y) \in T)$
$\equiv <\text{ax18b}> \exists (z . ((x,z) \in R \wedge (z,y) \in S) \vee \exists (z . (x,z) \in R \wedge (z,y) \in T)$
$\equiv <\text{def } \circ> (x,y) \in R \circ S \vee (x,y) \in R \circ T$
$\equiv <\text{axV5}> (x,y) \in (R \circ S) \cup (R \circ T)$
wegens veralgemening + axV2

Oefening 15:

$$\begin{aligned}
& R \circ (S \circ T) = (R \circ S) \circ T \\
(x,y) \in R \circ (S \circ T) \; & \equiv \langle \text{def } \circ \rangle \; \exists (z \,.\, (x,z) \in R \wedge (z,y) \in (S \circ T)) \\
& \equiv \langle \text{def } \circ \rangle \; \exists (z \,.\, (x,z) \in R \wedge \exists (v \,.\, (z,v) \in S \wedge (v,y) \in T)) \\
& \equiv \langle \text{st77} \rangle \; \exists (z \,.\, \exists (v \,.\, (x,z) \in R \wedge (z,v) \in S \wedge (v,y) \in T)) \\
& \equiv \langle \text{ax20b} \rangle \; \exists (v \,.\, \exists (z \,.\, (x,z) \in R \wedge (z,v) \in S \wedge (v,y) \in T)) \\
& \equiv \langle \text{st77} \rangle \; \exists (v \,.\, (v,y) \in T \wedge \exists (z \,.\, (x,z) \in R \wedge (z,v) \in S)) \\
& \equiv \langle \text{def } \circ \rangle \; \exists (v \,.\, (v,y) \in T \wedge (x,v) \in (R \circ S)) \\
& \equiv \langle \text{def } \circ \rangle \; (x,y) \in (R \circ S) \circ T
\end{aligned}$$

wegens veralgemening + axV2

Oefening 16: a:

$$\begin{aligned}
& \forall (x : X \,.\, (x,x) \in R) \equiv II_x \subseteq R \\
IIx \subseteq R \; & \equiv \langle \text{axV3} \rangle \; \forall ((x,y) \mid (x,y) \in II_x \,.\, (x,y) \in R) \\
& \equiv \langle \text{def } II_x \rangle \; \forall ((x,y) \mid x = y \,.\, (x,y) \in R) \\
& \equiv \langle \text{ax21} \rangle \; \forall (x \,.\, \forall (y \mid x = y \,.\, (x,y) \in R)) \\
& \equiv \langle \text{ax17a} \rangle \; \forall (x \,.\, (x,x) \in R)
\end{aligned}$$

Oefening 16: c:

$$\begin{aligned}
& \forall (x,y,z : X \,.\, xRy \wedge yRz \Rightarrow xRz) \equiv (R \circ R) \subseteq R \\
(R \circ R) \subseteq R \; & \equiv \langle \text{axV3} \rangle \; \forall ((x,y) \mid (x,y) \in R \circ R \,.\, (x,y) \in R) \\
& \equiv \langle \text{ax23} \rangle \; \forall ((x,y) \,.\, (x,y) \in R \circ R \Rightarrow (x,y) \in R) \\
& \equiv \langle \text{def } \circ \rangle \; \forall ((x,y) \,.\, \exists (z \,.\, (x,z) \in R \wedge (z,y) \in R) \Rightarrow (x,y) \in R) \\
& \equiv \langle \text{st36} \rangle \; \forall ((x,y) \,.\, \neg \exists (z \,.\, (x,z) \in R \wedge (z,y) \in R) \vee (x,y) \in R) \\
& \equiv \langle \text{st75c} \rangle \; \forall ((x,y) \,.\, \forall (z \,.\, \neg((x,z) \in R \wedge (z,y) \in R)) \vee (x,y) \in R) \\
& \equiv \langle \text{ax24} \rangle \; \forall ((x,y) \,.\, \forall (z \,.\, \neg((x,z) \in R \wedge (z,y) \in R) \vee (x,y) \in R)) \\
& \equiv \langle \text{ax21} \rangle \; \forall (x,y,z \,.\, \neg((x,z) \in R \wedge (z,y) \in R) \vee (x,y) \in R) \\
& \equiv \langle \text{st36} \rangle \; \forall (x,y,z \,.\, (x,z) \in R \wedge (z,y) \in R \Rightarrow (x,y) \in R)
\end{aligned}$$

Oefening 18: a:

$$\begin{aligned}
R \downarrow A \subseteq A \; & \equiv \langle \text{axV3} \rangle \; \forall (x \mid x \in R \downarrow A \,.\, x \in A) \\
& \equiv \langle \text{def } \downarrow \rangle \; \forall (x \mid \forall (y \,.\, yRx \Rightarrow y \in A) \,.\, x \in A) \\
& \equiv \langle \text{ax23} \rangle \; \forall (x \,.\, \forall (y \,.\, yRx \Rightarrow y \in A) \Rightarrow x \in A) \\
& \equiv \langle \text{R refl,st50} \rangle \; \forall (x \,.\, \forall (y \,.\, yRx \Rightarrow y \in A) \Rightarrow xRx \Rightarrow x \in A) \\
& \equiv \langle \text{st74} \rangle \; \forall (x \,.\, 1) \\
& \equiv \langle \text{st69} \rangle \; 1
\end{aligned}$$

Oefening 18: c:

$$\begin{aligned}
& R \uparrow (A \cup B) = R \uparrow A \cup R \uparrow B \\
x \in R \uparrow (A \cup B) \; & \equiv \langle \text{def } \uparrow \rangle \; \exists (y \,.\, yRx \wedge y \in A \cup B) \\
& \equiv \langle \text{axV5} \rangle \; \exists (y \,.\, yRx \wedge (y \in A \vee y \in B)) \\
& \equiv \langle \text{st28} \rangle \; \exists (y \,.\, (yRx \wedge y \in A) \vee (yRx \wedge y \in B)) \\
& \equiv \langle \text{ax18b} \rangle \; \exists (y \,.\, yRx \wedge y \in A) \vee \exists (y \,.\, yRx \wedge y \in B) \\
& \equiv \langle \text{def } \uparrow \rangle \; x \in R \uparrow A \vee x \in R \uparrow B
\end{aligned}$$

≡ <axV5> x ∈ (R ↑ A ∪ R ↑ B)
wegens veralgemening + axV2

<u>Oefening 18: i:</u>

R ↓ (R ↓ A) = R ↓ A
wegens veralgemening voldoende aan te tonen:
x ∈ R ↓ (R ↓ A) ≡ x ∈ (R ↓ A)
wegens “wederzijdse implicatie” voldoende om aan te tonen:
x ∈ R ↓ (R ↓ A) => x ∈ (R ↓ A)
en x ∈ R ↓ A => x ∈ R ↓ (R ↓ A)

x ∈ R ↓ (R ↓ A) => x ∈ (R ↓ A)
<= <st74> ∀ (x . x ∈ R ↓ (R ↓ A) => x ∈ R ↓ A)
≡ <ax23> ∀ (x | x ∈ R ↓ (R ↓ A) . x ∈ R ↓ A)
≡ <axV3> R ↓ (R ↓ A) ⊆ R ↓ A
≡ <oef18a> 1
Bewijs van de vorm: 1 =>p
≡ <st50> p
∀ (x . p) => <p => q> ∀ (x . q)
lemma: (p => q) => (∀ (x . p) => ∀ (x . q))
We nemen het antecedent p => q aan
∀ (x . p) ≡ <aanname,st21> ∀ (x . p ∧ (p => q))
≡ <st43> ∀ (x . p ∧ q)
≡ <ax18a> ∀ (x . p) ∧ ∀ (x . q)
=> <st53b> ∀ (x . q)

x ∈ R ↓ A ≡ <def ↓ > ∀ (z . zRx => z ∈ A)
≡ <R trans,st21> ∀ (z . (zRx => z ∈ A) ∧ (zRy ∧ yRx => zRx))
=> <lemma,st59a> ∀ (z . zRy ∧ yRx => z ∈ A)
≡ <st42> ∀ (z . yRx => zRy => z ∈ A)
≡ <st36> ∀ (z . ¬(yRx) ∨ (zRy => z ∈ A))
≡ <ax24> ¬(yRx) ∨ ∀ (z . zRy => z ∈ A)
≡ <st36> yRx => ∀ (z . zRy => z ∈ A)
≡ <def ↓ > yRx => y ∈ R ↓ A
we hebben aangetoond:
x ∈ R ↓ A => yRx => y ∈ R ↓ A
wegens veralgemening:
∀ (y . x ∈ R ↓ A => yRx => y ∈ R ↓ A)
≡ <2x st36, ax24> x ∈ R ↓ A => ∀ (y . yRx => y ∈ R ↓ A)
≡ <def ↓ > x ∈ R ↓ A => x ∈ R ↓ (R ↓ A)

<u>Oefening 19: a:</u>

x minimaal element van A
x ∈ A ∧ ∀ (y | y < x . y ∉ A)
b en c zijn zowel maximale als minimale elementen van A
x is het kleinste element van A

$x \in A \wedge \forall (y \mid y \in A . x \leq y)$
geen kleinste en geen grootste elementen

Oefening 19: b:

$A = \{b,c\}$
b en c, zowel minimale als maximale elementen

Oefening 21:

$\forall (x . xRx) \Rightarrow \forall (A . \neg\exists (b . b \in A \wedge \forall (x \mid xRb . x \notin A)))$
We nemen $\forall (x . xRx)$ aan en bewijzen
$\forall (A . \neg\exists (b . b \in A \wedge \forall (x \mid xRb . x \notin A)))$
Wegens veralgemening is het voldoende om aan te tonen:
$\neg\exists (b . b \in A \wedge \forall (x \mid xRb . x \notin A))$
$\equiv \langle \text{st75c} \rangle \forall (b . b \in A \wedge \forall (x \mid xRb . x \notin A))$
Wegens veralgemening is het voldoende om aan te tonen
$\neg(b \in A \wedge \forall (x \mid xRb . x \notin A))$

$$
\begin{aligned}
\neg(b \in A \wedge \forall (x \mid xRb . x \notin A)) \quad & \equiv \langle \text{st29a, def } \notin \rangle\ b \notin A \vee \neg\forall (x \mid xRb . x \notin A) \\
& \equiv \langle \text{ax25, def } \notin \rangle\ b \notin A \vee \exists (x \mid xRb . x \in A) \\
& \equiv \langle \text{st76a} \rangle\ b \notin A \vee \exists (x . xRb \wedge x \in A) \\
& \Leftarrow \langle \text{st84} \rangle\ b \notin A \vee (bRb \wedge b \in A) \\
& \equiv \langle \text{aanname} \rangle\ b \notin A \vee (1 \wedge b \in A) \\
& \equiv \langle \text{st21} \rangle\ b \notin A \vee b \in A \\
& \equiv \langle \text{ax11} \rangle\ 1 \\
& \qquad (1 \Rightarrow p) \equiv p
\end{aligned}
$$

$$
\begin{aligned}
\neg(b \in A \wedge \forall (x \mid xRb . x \notin A)) \quad & \equiv \langle \ldots \rangle\ b \notin A \vee \exists (x . xRb \wedge x \in A) \\
& \equiv \langle \text{ax11} \rangle\ b \notin A \vee \exists (x \mid x = b \vee \neg(x = b) . xRb \wedge x \in A) \\
& \equiv \langle \text{ax19b} \rangle\ b \notin A \vee \exists (x \mid x = b . xRb \wedge x \in A) \vee \\
& \qquad \exists (x \mid \neg(x = b) . xRb \wedge x \in A) \\
& \equiv \langle \text{ax17b} \rangle\ b \notin A \vee (bRb \wedge b \in A) \vee \\
& \qquad \exists (x \mid \neg(x = b) . xRb \wedge x \in A) \\
& \equiv \langle \ldots \rangle\ b \notin A \vee b \in A \vee \exists (x \ldots) \\
& \equiv \langle \ldots \rangle 1
\end{aligned}
$$

Oefening 22:

$b \text{ isklst}_R A \equiv b \in A \wedge \forall (x \mid x \in A . bRx)$ $\qquad (b \text{ isklst}_R A \equiv b \text{ ismin}_S A)$

$$
\begin{aligned}
b \text{ isklst}_R A \quad & \equiv \langle \text{def subst} \rangle\ b \in A \wedge \forall (x \mid x \in A . bRx) \\
& \equiv \langle \text{ax23} \rangle\ b \in A \wedge \forall (x . x \in A \Rightarrow bRx) \\
& \equiv \langle \text{st38,def } \notin \rangle\ b \in A \wedge \forall (x . \neg(bRx) \Rightarrow x \notin A) \\
& \equiv \langle \text{def S} \rangle\ b \in A \wedge \forall (x . \neg\neg(xSb) \Rightarrow x \notin A) \\
& \equiv \langle \text{st6} \rangle\ b \in A \wedge \forall (x . xSb \Rightarrow x \notin A) \\
& \equiv \langle \text{ax23} \rangle\ b \in A \wedge \forall (x \mid xSb . x \notin A) \\
& \equiv \langle \text{def ismin} \rangle\ b \text{ ismin}_S A
\end{aligned}
$$

Extra opgave:

Bewijs dat indien R antisymmetrisch is, elke deelverzameling van x hoogstens 1R kleinste element heeft.
b is een R kleinste element van A indien:
$b \in A \wedge \forall (x \mid x \in A . bRx)$
Opl: $b_1 \in A \wedge \forall (x \mid x \in A . b_1Rx) \wedge b_2 \in A \wedge \forall (x \mid x \in A . b_2Rx) \Rightarrow b_1 = b_2$

$b_1 \in A \wedge \forall (x \mid x \in A . b_1Rx) \wedge b_2 \in A \wedge \forall (x \mid x \in A . b_2Rx) \Rightarrow b_1 = b_2$
≡ <st42> $b_1 \in A . b_2 \in A \Rightarrow \forall (x \mid x \in A . b_1Rx) \wedge \forall (x \mid x \in A . b_2Rx) \Rightarrow b_1 = b_2$
we nemen $b_1 \in A$ en $b_2 \in A$ aan.
$p_1 \wedge p_2$
=> <$p_1 \Rightarrow q_1$, $p_2 \Rightarrow q_2$ > $q_1 \wedge q_2$

$\forall (x \mid x \in A . b_1Rx) \wedge \forall (x \mid x \in A . b_2Rx)$
≡ <ax23> $\forall (x . x \in A \Rightarrow b_1Rx) \wedge \forall (x . x \in A \Rightarrow b_2Rx)$
=> <st74, lemma> $(b_2 \in A \Rightarrow b_1Rb_2) \wedge (b_1 \in A \Rightarrow b_2Rb_1)$
lemma $p \wedge q \Rightarrow (p \Rightarrow r) \Rightarrow r \wedge q$
$p \wedge q \Rightarrow (q \Rightarrow r) \Rightarrow p \wedge r$

$p \wedge q \wedge (p \Rightarrow r) \Rightarrow r \wedge q$

$p \wedge q \wedge (p \Rightarrow r) \Rightarrow r \wedge q$
≡ <2x st42> $p \Rightarrow r \Rightarrow p \wedge q \Rightarrow r \wedge q$
we nemen $p \Rightarrow r$ aan
$p \wedge q \Rightarrow r \wedge q$ ≡ <st37> $p \wedge q \wedge r \wedge q \equiv p \wedge q$
≡ <st23> $p \wedge q \wedge r \equiv p \wedge q$
≡ <st31> $(p \wedge r \equiv p) \wedge q \equiv q$
≡ <st37> $(p \Rightarrow r) \wedge q \equiv q$
≡ <aanname> $1 \wedge q \equiv q$
≡ <st21> $q \equiv q$
→ instantiatie
≡ <aanname> $(1 \Rightarrow b_1Rb_2) \wedge (1 \Rightarrow b_2Rb_1)$
≡ <st50> $b_1Rb_2 \wedge b_2Rb_1$
=> <R antisymm> $b_1 = b_2$

Oefening 23:

Pn ≡ in een groep van n mensen heeft iedereen rood haar

$P0 \wedge \forall (n \mid 2 \leq n . Pn \Rightarrow P(n + 1))$
$P2 \wedge \forall (n \mid 2 \leq n . Pn \Rightarrow P(n + 1)) \equiv \forall (n . Pn)$
$P0 \wedge \forall (n . Pn \Rightarrow P(n + 1)) \equiv \forall (n . Pn)$

Oefening 24:

$Pn \equiv \exists(k \mid k \in N \,.\, \exists(l \mid l \in N \,.\, n = 2k + 5l))$
$\forall(n \mid 4 \leq n \,.\, Pn) \equiv P4 \wedge \forall(n \mid 4 \leq n \,.\, Pn \Rightarrow P(n + 1))$

Basisregel:

$P4 \equiv \text{<def P>}\ \exists(k \mid k \in N \,.\, \exists(l \mid l \in N \,.\, 4 = 2k + 5l))$
$\equiv \text{<st76>}\ \exists(k \,.\, k \in N \wedge \exists(l \,.\, l \in N \wedge 4 = 2k + 5l))$
$\Leftarrow \text{<st84>}\ 2 \in N \wedge \exists(l \,.\, l \in N \wedge 4 = 2.2 + 5l)$
$\equiv \text{<rekenen,st21>}\ \exists(l \,.\, l \in N \wedge 4 = 4 + 5l)$
$\equiv \text{<st24>}\ 0 \in N \wedge 4 = 4 + 5.0)$
$\equiv \text{<rekenen>}\ 1$

Inductiestap:

$\forall(n \mid 4 \leq n \,.\, Pn \Rightarrow P(n + 1))$
$\equiv \text{<ax23>}\ \forall(n \,.\, 4 \leq n \Rightarrow Pn \Rightarrow P(n + 1))$
wegens de metastelling “veralgemening” is het voldoende om aan te tonen:
$4 \leq n \Rightarrow Pn \Rightarrow P(n + 1)$
we nemen $4 \leq n$ aan:
$Pn \Rightarrow P(n + 1)$
$\equiv \text{<def P>}\ \exists(k \mid k \in N \,.\, \exists(l \mid l \in N \,.\, n = 2k + 5l)) \Rightarrow P(n + 1)$
wegens de metastelling “getuige” is het voldoende om aan te tonen:
$k \in N \wedge \exists(l \mid l \in N \,.\, n = 2k + 5l) \Rightarrow P(n + 1)$
$\equiv \text{<st42>}\ k \in N \Rightarrow \exists(l \mid l \in N \,.\, n = 2k + 5l) \Rightarrow P(n + 1)$
we nemen $k \in N$ aan en bewijzen:
$\exists(l \mid l \in N \,.\, n = 2k + 5l) \Rightarrow P(n + 1)$
wegens metastelling “getuige” voldoende om aan te tonen:
$l \in N \wedge n = 2k + 5l \Rightarrow P(n + 1)$
$\equiv \text{<st42>}\ l \in N \Rightarrow n = 2k + 5l \Rightarrow P(n + 1)$
we nemen $l \in N$ aan en bewijzen
$n = 2k + 5l \Rightarrow P(n + 1)$
we gebruiken gevallenonderzoek en bewijzen
(1) $l = 0 \vee l \geq 1$
(2) $l = 0 \Rightarrow n = 2k + 5l \Rightarrow P(n + 1)$
(3) $l \geq 1 \Rightarrow n = 2k + 5l \Rightarrow P(n + 1)$

(1) onmiddellijk voldaan wegens aanname $l \in N$

(2) we nemen $l = 0$ aan

$n = 2k + 5l$
$\equiv \text{<aanname l = 0, rekenen>}\ n = 2k$
$\equiv \text{<aanname n} \geq \text{4, rekenen>}\ k - 2 \in N \wedge n = 2k$
$\equiv \text{<rekenen>}\ k - 2 \in N \wedge n + 1 = 2(k - 2) + 5.1$
$\Rightarrow \text{<st84>}\ \exists(k' \,.\, k' \in N \wedge n + 1 = 2k' + 5.1)$
$\equiv \text{<st84,lemma,1} \in \text{N>}\ \exists(k' \,.\, k' \in N \wedge \exists(l' \,.\, l' \in N \wedge n + 1 = 2k' + 5.l')$
$\equiv \text{<st76>}\ \exists(k' \,.\, k' \in N \,.\, \exists(l' \mid l' \in N \,.\, n + 1 = 2k' + 5.l')$
$\equiv \text{<def P>}\ P(n + 1)$

(3) we nemen $l \geq 1$ aan

$n = 2k + 5l \quad \equiv \text{<aanname l} \geq \text{1, rekenen>}\ l - 1 \in N \wedge n = 2k + 5l)$

≡ <rekenen> $l - 1 \in N \wedge n + 1 = 2k + 5l + 1)$

=> <st84> $\exists (k' . k' \in N \wedge l - 1 \in N \wedge n + 1 = 2k' + 5l)$

≡ <lemma,st84> $\exists (k' . k' \in N \wedge \exists (l' . l' \in N \wedge n + 1 = 2k' + 5l')$

≡ <st76> $\exists (k' . k' \in N . \exists (l' \mid l' \in N . n + 1 = 2k' + 5l')$

≡ <def P> $P(n + 1)$

lemma $(p \Rightarrow q) \Rightarrow \exists (x . p \wedge r) \Rightarrow \exists (x . q \wedge r)$

bewijs als oefening (metastelling getuige)

+ eerder bewezen lemma