

oefening 1.9.19.

1) gegeven: f is holomorf in $\bar{B}(z_0, R)$

$$M_R := \max_{|z-z_0|=R} |f(z)|$$

t.b. $|f^{(n)}(z_0)| \leq n! \frac{M_R}{R^n} \quad (n = 0, 1, 2, \dots)$.

Bewijs

f is holomorf op $\bar{B}(z_0, R) \xrightarrow{1.9.7.} \exists a_n \in \mathbb{C},$

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-z_0)^n \quad (z \in B(z_0, R))$$

met $a_n = \frac{f^{(n)}(z_0)}{n!}$

en $a_n = \frac{1}{2\pi i} \int_{\partial B_+(z_0, r)} \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi.$

Zig $r \rightarrow R^-$:

$$a_n = \frac{1}{2\pi i} \lim_{r \rightarrow R^-} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} r i e^{i\theta} d\theta$$

Omdat f continu is op $\bar{B}(z_0, R)$

$$\Rightarrow \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} r i e^{i\theta} \xrightarrow{r \rightarrow R^-} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n+1}} R i e^{i\theta}$$

$$\Rightarrow a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n+1}} R i e^{i\theta} d\theta$$

$$\Rightarrow |a_n| \leq \frac{1}{2\pi i} \int_0^{2\pi} \frac{|f(z_0 + Re^{i\theta})|}{R^n} d\theta$$

$$\leq \frac{M_R}{R^n}$$

$$\Rightarrow \frac{|f^{(n)}(z_0)|}{n!} \leq \frac{M_R}{R^n}$$

$$\Leftrightarrow |f^{(n)}(z_0)| \leq n! \frac{M_R}{R^n}$$

2) tov: $f_\lambda(x) = \sin(\lambda x)$, $f'(0) = \lambda$
neem $0 < \lambda$.

1.9.20.

1) $\Rightarrow$ 2):

mit 1) ($\exists R > 0$) zodat $f \equiv g$ en g is holomorf op $B(x_0, R)$

Zij $|x - x_0| < \frac{R}{3} \Rightarrow g$ is holomorf op $\bar{B}(x, \frac{R}{3}) \subseteq B(x_0, R)$

$$\stackrel{=}{\Rightarrow} \underset{\text{(ongelijkheid van Cauchy)}}{|g^{(n)}(x)|} \leq n! \frac{M_{R/3}}{(\frac{R}{3})^n}$$

$$\leq n! \frac{M_{2R/3}}{(\frac{R}{3})^n} \leq \frac{C n!}{(\frac{R}{3})^n}$$

Door extremumstelling van Weierstrass is $|g|$ begrensd op $\bar{B}(x_0, 2R)$

2) $\Rightarrow$ 3): Het volstaat om aan te tonen dat

$$\frac{|f^{(n+1)}(\xi)(x-x_0)^{n+1}|}{(n+1)!} \rightarrow 0 \text{ als } n \rightarrow \infty$$

$$(n+1)!$$

(waar x voldoende dicht

bij x_0)

$$\Rightarrow \left| \frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!} \right| \stackrel{(2)}{\leq} \frac{C A^{n+1} (n+1)! (x-x_0)^{n+1}}{(n+1)!}$$

$$\rightarrow 0 \text{ als } |x-x_0| < \frac{1}{A}$$

DEFENING 2.4.11

$$1) f(z) = \frac{e^{iz}}{1+z^2}$$

- geïsoleerde punten bepalen: de nulpunten van de noemer:

$$1+z^2 = 0 \quad (\Rightarrow) \quad z^2 = -1 \quad (\Rightarrow) \quad z = \pm i \quad \checkmark$$

- mult $f(z) = -1 \rightarrow$ enkelvoudige polen $\checkmark$
 $z = \pm i$

- residu bepalen:

$$\begin{aligned} \text{res } f(z) &= \lim_{z \rightarrow i} \frac{(z-i) e^{iz}}{(z-i)(z+i)} = \lim_{z \rightarrow i} \frac{e^{iz}}{z+i} \\ &= \frac{e^{i^2}}{i+i} = \frac{e^{-1}}{2i} = \frac{e^{-1}}{2i} \cdot \frac{i}{i} = \frac{-i}{2e} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{res } f(z) &= \lim_{z \rightarrow -i} \frac{(z+i) e^{iz}}{(z-i)(z+i)} = \lim_{z \rightarrow -i} \frac{e^{iz}}{z-i} \\ &= \frac{e^{-i^2}}{-i-i} = \frac{e}{-2i} \cdot \frac{i}{i} = \frac{ei}{2} \quad \checkmark \end{aligned}$$

$$2) f(z) = \frac{\sin(\pi z)}{(z-1)^3}$$

- geïsoleerde singuliere punten bepalen: nulpunten van de noemer:

$$(z-1)^3 = 0 \quad (\Rightarrow) \quad z = 1 \quad \checkmark \quad (3x)$$

$$\begin{aligned} \text{mult } f(z) &= -\text{mult } (z-1)^3 + \text{mult } \sin(\pi z) \\ &= -3 + 1 = -2 \rightarrow \text{dubbele pool} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \sin(\pi) &= 0 \\ \sin'(\pi) &= \cos \pi = -1 \neq 0 \end{aligned}$$

- residu bepalen:

$$\begin{aligned} \text{res } f(z) &= \frac{1}{1!} \lim_{z \rightarrow 1} \left(\frac{(z-1)^2 \sin(\pi z)}{(z-1)^3} \right)' \\ &= \lim_{z \rightarrow 1} \left(\frac{\sin(\pi z)}{z-1} \right)' \\ &= \lim_{z \rightarrow 1} \frac{(z-1)\pi \cos(\pi z) - \sin(\pi z)}{(z-1)^2} \\ &= \frac{0}{0} \stackrel{H}{=} \lim_{z \rightarrow 1} \frac{\pi \cos(\pi z) - (z-1)\pi^2 \sin(\pi z)}{2(z-1)} \\ &= \lim_{z \rightarrow 1} \frac{-\pi^2 \sin(\pi z)}{2} \\ &= 0 \quad \checkmark \end{aligned}$$

3) $f(z) = \frac{1+z^{10}}{z^6(z^2+4)}$

- geïsoleerde singuliere punten bepalen:

$$z^6(z^2+4) = 0 \quad (\Rightarrow) \quad z^6 = 0 \vee z^2+4=0$$

$$\Rightarrow z=0 \vee z^2 = -4$$

$(=) \quad z = \pm 2i \quad \checkmark$

- multipliciteiten bepalen:

mult $f(z) = -6 \rightarrow$ pol has order 6
 $z=0$

mult $f(z) = -1 \rightarrow$ enkelvoudige pool

mult $f(z) = -1$ $z = -2i$

- residue's behavior:

noch $z = 0$: res $f(z) = \frac{1}{5!} \lim_{z \rightarrow 0} \left(\frac{z^6 (1+z^4)^{(5)}}{z^6 (z^2 + 1)} \right)$
 $= \frac{1}{5!} \lim_{z \rightarrow 0} \left(\frac{1+z^4}{z^2 + 1} \right)^{(5)} = \dots = 0$ ✓

$$\begin{aligned} \text{und } z = -2i: \text{ res } f(z) &= \frac{1}{0!} \lim_{z \rightarrow -2i} (z-2i)(z+2i) \\ &= \lim_{z \rightarrow -2i} \frac{1+z^{10}}{z^6(z+2i)} = \frac{1+(2i)^{10}}{(2i)^6(2i+2i)} \\ &= \frac{1-2^{10}}{-64(i)} = \frac{-1023}{-256i} \cdot \frac{i}{i} = -\frac{1023i}{256} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{von } z = -2i: \text{ res } f(z) &= \lim_{z \rightarrow -2i} \frac{(z+2i)(1+z^{10})}{z^6(z+2i)(z-2i)} \\ &= \lim_{z \rightarrow -2i} \frac{1+z^{10}}{z^6(z-2i)} = \frac{1+(-2i)^{10}}{(2i)^6(-4i)} \\ &= \frac{-1023}{+265i} \cdot \frac{i}{i} = \frac{-1023i}{265} \quad \checkmark \end{aligned}$$

$$4) \sin\left(\frac{z}{z+1}\right) = f(z)$$

- geïsoleerde punten: nulpunten van $z+1$:

$$z = -1 \quad \checkmark$$

- type bepalen: (multipliciteit met reeksreeks afleesbaar):

→ pool? Neen, want $\lim_{z \rightarrow -1} f(z)$ bestaat niet

↳ opdracht 2.4.5.

→ ophelbaar? Neen, want de limiet $\notin \mathbb{C}$

→ essentieel div $\checkmark$

- residu bepalen:

via Laurentontwikkeling met $z = -1$ als singulier punt:

$$\sin\left(\frac{z}{z+1}\right) = \sin\left(1 - \frac{1}{z+1}\right)$$

Om $\frac{1}{z+1}$ af te zonderen,

andere krijg je oneindige som voor residu

$$= \sin(1) \cos\left(\frac{1}{z+1}\right) - \cos(1) \sin\left(\frac{1}{z+1}\right)$$

$$= \sin(1) \left(1 - \frac{1}{2!(z+1)^2} + \frac{1}{4!(z+1)^4} - \dots\right)$$

$$- \cos(1) \left(\frac{1}{z+1} - \frac{1}{3!(z+1)^3} + \frac{1}{5!(z+1)^5} - \dots\right)$$

↳ heeft oneindig veel $n \in \mathbb{N}$ waarvoor $a_n \neq 0$
 $\Rightarrow z = -1$ is inderdaad essentieel singulier

$$\rightarrow \operatorname{res}_{z=-1} \left(\sin\left(\frac{z}{z+1}\right) \right) = -\cos(1) \quad \checkmark$$

OF via integraal:

$$\operatorname{res}_{z=-1} \left(\sin\left(\frac{z}{z+1}\right) \right) = \frac{1}{2\pi i} \int_{\partial B_{+}(-1, \varepsilon)} \sin\left(\frac{z}{z+1}\right) dz$$

$\varepsilon > 0$, klein genoeg

residuformule op het variërende toepassen ipv minnensce



$$= \frac{1}{2\pi i} \int_{\partial B_{+}(-1, \varepsilon)} \sin\left(\frac{z}{z+1}\right) dz$$

$$\stackrel{(u=1/2)}{=} \frac{1}{2\pi i} \int_{\partial B_{-}(0, \frac{1}{\varepsilon})} \sin\left(\left(\frac{z}{u} - 1\right)u\right) \left(\frac{-du}{u^2}\right)$$

$$= \frac{1}{2\pi i} \int_{\partial B_{+}(0, 1/\varepsilon)} \frac{\sin(1-u)}{u^2} du \quad u=0 \text{ is pool van orde 2!!}$$

$$= \operatorname{res}_{u=0} \frac{\sin(1-u)}{u^2} = \lim_{u \rightarrow 0} \frac{d}{du} \sin(1-u) = -\cos(1) \quad \checkmark$$

$$5) f(z) = \frac{e^z}{\sin^2(z)}$$

- geadeerde punten bepalen:

$$\sin^2(z) = 0 \Rightarrow \sin(z) = 0$$

$$\Rightarrow z = k\pi, k \in \mathbb{Z} \quad \checkmark$$

- mult $\frac{e^z}{\sin^2(z)} = -2 \rightarrow$ dubbele polen $\checkmark$

- residu's bepalen:

$$\text{res}_{z=k\pi} f(z) = \lim_{z \rightarrow k\pi} \left(\frac{(z-k\pi)e^z}{\sin^2(z)} \right)$$

$$= \lim_{z \rightarrow k\pi} \frac{\sin^2(z)[e^z + (z-k\pi)e^z] - (z-k\pi)e^z \sin(2z)}{\sin^4(z)}$$

$$= \frac{0}{0} \stackrel{\text{L'H}}{=} \lim_{z \rightarrow k\pi} \frac{2\sin(2z)[e^z + (z-k\pi)e^z] + \sin^2(z)[e^z + (z-k\pi)e^z]' - (e^z + (z-k\pi)e^z)\sin(2z) - (z-k\pi)e^z 2\cos(2z)}{4\sin^3(z)\cos(z)}$$

$$= \frac{0}{0} = \dots = e^{k\pi} \text{ als } k \neq 0$$

$$= 0 \text{ als } k = 0$$

$$6) f(z) = \frac{1}{\sin(z^2)}$$

- geadeerde punten: $z^2 = k\pi, k \in \mathbb{Z}$

$$\Rightarrow z = \pm \sqrt{k\pi}, k \in \mathbb{N}$$

$$\vee z = \pm \sqrt{k\pi}i, k \in \mathbb{Z} \setminus \mathbb{N}$$

- mult $f(z) = -1 \rightarrow$ enkelvoudige polen

- mult $f(z) = -1 \rightarrow$ " " " "

- residu's bepalen:

voor $z = \sqrt{k\pi}, k \in \mathbb{N}$

$$\text{res}_{z=\sqrt{k\pi}} f(z) = \lim_{z \rightarrow \sqrt{k\pi}} \frac{z - \sqrt{k\pi}}{\sin(z^2)} = \frac{0}{0}$$

$$\stackrel{\text{L'H}}{=} \lim_{z \rightarrow \sqrt{k\pi}} \frac{1}{2\cos(z^2) \cdot 2z} = \frac{(-1)^k}{2\sqrt{k\pi}} \quad \checkmark$$

voor $z = -\sqrt{k\pi}, k \in \mathbb{N}$

$$\text{res}_{z=-\sqrt{k\pi}} f(z) = \lim_{z \rightarrow -\sqrt{k\pi}} \frac{z + \sqrt{k\pi}}{\sin(z^2)} = \frac{0}{0}$$

$$\stackrel{\text{L'H}}{=} \lim_{z \rightarrow -\sqrt{k\pi}} \frac{1}{2\cos(z^2) \cdot 2z} = \frac{-(-1)^k}{2\sqrt{k\pi}} \quad \checkmark$$

$$\text{voor } z = \sqrt{k\pi}i, \quad k \in \mathbb{Z} \setminus \{0\}$$

$$\begin{aligned} \operatorname{res}_{z=\sqrt{k\pi}i} f(z) &= \lim_{z \rightarrow \sqrt{k\pi}i} \frac{z - \sqrt{k\pi}i}{\sin(z^2)} \\ &= \dots = \frac{-(-1)^k}{2\sqrt{k\pi}}; \quad \checkmark \end{aligned}$$

$$\text{voor } z = -\sqrt{k\pi}i, \quad k \in \mathbb{Z} \setminus \{0\}$$

$$\begin{aligned} \operatorname{res}_{z=-\sqrt{k\pi}i} f(z) &= \lim_{z \rightarrow -\sqrt{k\pi}i} \frac{z + \sqrt{k\pi}i}{\sin(z^2)} \\ &= \frac{(-1)^k}{2\sqrt{k\pi}}; \quad \checkmark \end{aligned}$$

$$7) f(z) = \frac{z+2}{1-\cos z}$$

- gelijkende punten:

$$\cos z = 1 \quad (\Leftrightarrow) \quad z = 2k\pi \quad k \in \mathbb{Z}$$

$$\text{- mult}_{z=2k\pi} f(z) = -\text{mult}_{z=2k\pi} (1-\cos z) = -2$$

$$\Leftrightarrow 1 - \cos(2\pi k) = 0$$

$$(1 - \cos(2\pi k))' = \sin(2\pi k) = 0$$

$$(\sin(2k\pi))' = \cos(2k\pi) \neq 0$$

$\rightarrow$ dubbele pool

- residu bepalen

$$\operatorname{res}_{z=2k\pi} f(z) = \lim_{z \rightarrow 2k\pi} \left(\frac{(z-2k\pi)^2 (1-\cos z)}{1-\cos z} \right)'$$

$$= \lim_{z \rightarrow 2k\pi} \left(\frac{-z^2 + 2k\pi z + (1-2k\pi)}{1-\cos z} \right)'$$

$$= \lim_{z \rightarrow 2k\pi} \frac{(1-\cos z)(-2z + 2k\pi) - (-z^2 + 2k\pi z + (1-2k\pi)) \sin z}{(1-\cos z)^2}$$

$$= \frac{0}{0} \quad \Leftrightarrow$$

$$\lim_{z \rightarrow 2k\pi} \frac{(1-\cos z) \cdot (-2) + (-2z + 2k\pi)(\sin z) - \cos z(-z^2 + 2k\pi z + (1-2k\pi))}{2(1-\cos z)\sin z}$$

$$= \lim_{z \rightarrow 2k\pi} \frac{(1-\cos z)(-2) - \cos z(-z^2 + 2k\pi z + (1-2k\pi))}{2(1-\cos z)\sin z}$$

$$= \dots = 2$$

$$8) f(z) = \frac{z^{2n}}{(z-1)^n}$$

$$- (z-1)^n = 0 \quad (\Rightarrow) z=1$$

- multipliciteit: mult $f(z) = -n \rightarrow$ pool van $2n$ en n

(als $n=0 \Rightarrow$ geen geïsoleerde punten: $f(z)=1$)

- residu:

$$\operatorname{res}_{z=1} f(z) = \frac{1}{n!} \lim_{z \rightarrow 1} \left(\frac{(z-1)^n z^{2n}}{(z-1)^n} \right)^{(n-1)}$$

$$= \frac{1}{n!} \lim_{z \rightarrow 1} (z^{2n})^{(n-1)}$$

$$= \frac{(2n) \cdot (2n-1) \cdot \dots \cdot (2n-(n-1)+1)}{n!} \lim_{z \rightarrow 1} z^{2n-n}$$

$$= \frac{(2n)!}{(n-1)!} \quad \checkmark$$

OEFENING 2.4.12

$$1) \quad f(z) = \frac{4-3z}{z(z-1)(z-2)}$$

gevraagd: $\int_{\partial B_+(0, 3/2)} f(z) dz$

- geïsoleerde punten van $f(z)$ in $\overline{B}(0, 3/2)$

$$z(z-1)(z-2) = 0 \Leftrightarrow z=0 \vee z=1 \vee \cancel{z=2} \quad \rightarrow \text{niet nodig}$$

- mult $f(z)$ = mult $f(z)$ = mult $f(z)$ = -1
 $z=0$ $z=1$ $z=2$

$\rightarrow$ allemaal enkelvoudige polen $\rightarrow$

- residu's bepalen:

voor $z=0$:

$$\begin{aligned} \text{res}_{z=0} f(z) &= \lim_{z \rightarrow 0} \frac{z(4-3z)}{z(z-1)(z-2)} \\ &= \lim_{z \rightarrow 0} \frac{(4-3z)}{(z-1)(z-2)} \\ &= \frac{4}{(-1)(-2)} = 2 \end{aligned}$$

voor $z=1$

$$\begin{aligned} \text{res}_{z=1} f(z) &= \lim_{z \rightarrow 1} \frac{(z-1)(4-3z)}{z(z-1)(z-2)} \\ &= \lim_{z \rightarrow 1} \frac{(4-3z)}{z(z-2)} \\ &= \frac{1}{1(-1)} = -1 \end{aligned}$$

- integraal berekenen via residu'sstelling:

$$\int_{\partial B_+(0, 3/2)} f(z) = 2\pi i (2-1) = 2\pi i \quad \checkmark$$

$$2) \quad f(z) = \frac{e^{z^2}}{z^2(z^2+2z+2)}$$

gevraagd: $\int_{\partial B_+(0, 3)} f(z) dz$

- geïsoleerde punten van $f(z)$ in $\overline{B}(0, 3)$:

$$\begin{aligned} z^2(z^2+2z+2) &= 0 \Leftrightarrow z=0 \vee z^2+2z+2=0 \\ D &= 4-4 \cdot 2 = -4 \\ z_{1,2} &= \frac{-2 \pm 2i}{2} \quad \begin{matrix} \swarrow -1+i \\ \searrow -1-i \end{matrix} \end{aligned}$$

- mult $f(z)$ = -2 $\rightarrow$ dubbele pool
 $z=0$

- mult $f(z)$ = -1 $\rightarrow$ enkelvoudige pool
 $z = -1 \pm i$

- residu's berekenen

voor $z=0$

$$\text{res } f(z) = \lim_{z \rightarrow 0} \left(\frac{z^2 \cdot e^{zt}}{z^2(z^2+2z+2)} \right)'$$

$$= \lim_{z \rightarrow 0} \left(\frac{e^{zt}}{z^2+2z+2} \right)'$$

$$= \lim_{z \rightarrow 0} \frac{te^{zt}(z^2+2z+2) - (2z+2)e^{zt}}{(z^2+2z+2)^2}$$

$$= \frac{t \cdot e^0(2) - (2)e^0}{(2)^2} = \frac{t-1}{2}$$

voor $z = -1+i$

$$\text{res } f(z) = \lim_{z \rightarrow -1+i} \frac{(z+1-i)e^{zt}}{(z+1+i)(z+1-i)z^2}$$

$$= \lim_{z \rightarrow -1+i} \frac{e^{zt}}{(z+1+i)z^2}$$

$$= \frac{e^{(-1+i)t}}{(-1+i+1+i)(-1+i)^2} = \frac{e^{(-1+i)t}}{(2i)(1-2i-1)}$$

$$= \frac{e^{(-1+i)t}}{(2i)(-2i)} = \frac{e^{(-1+i)t}}{-(-4(-1))} = \frac{e^{(-1+i)t}}{4}$$

voor $z = -1-i$

$$\text{res } f(z) = \lim_{z \rightarrow -1-i} \frac{(z+1+i)e^{zt}}{(z+1-i)(z+1+i)z^2}$$

$$= \lim_{z \rightarrow -1-i} \frac{e^{zt}}{(z+1-i)z^2} = \frac{e^{(-1-i)t}}{(-1-i+1-i)(-1-i)^2}$$

$$= \frac{e^{(-1-i)t}}{(-2i)(1+2i-1)} = \frac{e^{(-1-i)t}}{(-2i)(2i)}$$

$$= \frac{e^{(-1-i)t}}{4}$$

- integraal oplossen via residu'selling:

$$\int_{\partial B_+(0,3)} f(z) dz = 2\pi i \left[\frac{e^{(-1-i)t}}{4} + \frac{e^{(-1+i)t}}{4} + \frac{t-1}{2} \right]$$

$$= \pi i \left[\frac{e^{-t-it} + e^{-t+it}}{2} + t-1 \right]$$

$$= \pi i \left[e^{-t} \left(\frac{e^{-it} + e^{it}}{2} \right) + t-1 \right]$$

$$= \pi i \left[e^{-t} \cos t + t-1 \right] \checkmark$$

$$5) f(z) = \frac{\sin^6(z)}{(z - \frac{\pi}{6})^3}$$

gevraagd: $\int_{\partial B_+(0,1)} f(z) dz$

- geïsoleerde punten van $f(z)$ in $\bar{B}(0,1)$:

$$(z - \frac{\pi}{6})^3 = 0 \Rightarrow z = \frac{\pi}{6} \in \bar{B}(0,1)$$

- mult $f(z) = -3 \rightarrow$ pool van orde 3
 $z = \frac{\pi}{6}$

- residu bepalen:

$$\text{res}_{z=\frac{\pi}{6}} f(z) = \frac{1}{2!} \lim_{z \rightarrow \frac{\pi}{6}} \left(\frac{(z - \frac{\pi}{6})^3 \sin^6(z)}{(z - \frac{\pi}{6})^3} \right)^{(2)}$$

$$= \frac{1}{2} \lim_{z \rightarrow \frac{\pi}{6}} (6 \sin^5(z) \cos z)$$

$$= \frac{1}{2} \lim_{z \rightarrow \frac{\pi}{6}} (30 \sin^4(z) \cos^2 z - 6 \sin^6 z)$$

$$= \frac{1}{2} [30 \sin^4(\frac{\pi}{6}) \cos^2(\frac{\pi}{6}) - 6 \sin^6(\frac{\pi}{6})]$$

$$= \frac{1}{2} [30 (\frac{1}{2})^4 (\frac{\sqrt{3}}{2})^2 - 6 (\frac{1}{2})^6]$$

$$= 2(15 (\frac{1}{2})^4 \frac{3}{4} - 3 (\frac{1}{2})^6)$$

$$= 2(3 (\frac{1}{2})^4 [\frac{15}{4} - \frac{1}{4}]) = \frac{2 \cdot 3}{16} (\frac{14}{4}) = \frac{21}{16}$$

- integraal berekenen via residustelling:

$$\int_{\partial B_+(0,1)} f(z) dz = 2\pi i \frac{21}{16} = \frac{\pi i \cdot 21}{8}$$

oefening 2.5.2.

$$2) I = \int_0^{2\pi} \frac{dx}{(1+a\cos x)^2} \quad (0 < a < 1)$$

→ we zitten met een rationale functie van goniometrische functies → omvormen !!

$$f(x) = \frac{1}{(1+a\cos x)^2} = \frac{1}{\left[1 + \frac{a}{2}(e^{ix} + e^{-ix})\right]^2}$$

$$= \frac{1}{\left[\frac{1}{2}(2 + a(e^{ix} + e^{-ix}))\right]^2}$$

$$\text{stel } z = e^{ix} \Rightarrow -i \ln z = x$$

$$\text{zodat } f(z) = \frac{1}{(2 + a(z + 1/z))^2}$$

$$= \frac{1}{(2 + a(\frac{z^2 + 1}{z}))^2} = \frac{4z^2}{(2z + az^2 + a)^2}$$

en

$$I = \int_{\partial B(0,1)} \frac{(-i)4z^2}{(2z + az^2 + a)^2} dz, \text{ wegens } -i \frac{dz}{z} = dx.$$

→ we berekenen nu deze integraal met de residuïstelling:

→ geademde singuliere punten van $f(z)$:

$$(2z + az^2 + a)^2 = 0 \Leftrightarrow 2z + az^2 + a = 0 \text{ (2 keer)}$$

$$D = 4 - 4a^2 = 4(1-a^2)$$

$$z_{1,2} = \frac{-2 \pm 2\sqrt{1-a^2}}{2a} = \frac{-1 \pm \sqrt{1-a^2}}{a}$$

$$\text{zodat } z_1 = \frac{-1}{a} + \frac{\sqrt{1-a^2}}{a}$$

$$z_2 = \frac{-1}{a} - \frac{\sqrt{1-a^2}}{a} \rightarrow \notin \bar{B}(0,1) \text{ niet nodig!}$$

$$\text{en } 2z + az^2 + a = a(z + \frac{1}{a} + \frac{\sqrt{1-a^2}}{a})(z + \frac{1}{a} - \frac{\sqrt{1-a^2}}{a})$$

$$\hookrightarrow \text{mult}_{z=z_1} f(z) = -2 \rightarrow \text{dubbele pool}$$

$$\hookrightarrow \text{mult}_{z=z_2} f(z) = -2 \rightarrow \text{dubbele pool}$$

→ residu's bepalen

$$\text{res}_{z=z_1} f(z) = 1 - \lim_{z \rightarrow z_1} \left(\frac{-i4z^2}{a^2(z + \frac{1}{a} + \frac{\sqrt{1-a^2}}{a})^2} \right)'$$

~~$$= \lim_{z \rightarrow z_1} \left(\frac{-i4z^2}{a^2(z + \frac{1}{a} + \frac{\sqrt{1-a^2}}{a})^2} \right)'$$~~

~~$$= \lim_{z \rightarrow z_1} \frac{(-i4z^2(a^2(z - z_2)^2) + 4iz a^2 2(z - z_2))}{[a^2(z - z_2)^2]^2}$$~~

$$= \lim_{z \rightarrow z_1} \frac{2i4(-(z-z_2)^2 + 2(z-z_2)z)}{a^4(z-z_2)^4}$$

$$= \lim_{z \rightarrow z_1} \frac{i4(\cancel{z-z_2})[2z - (z-z_2)]}{a^4(z-z_2)^3}$$

$$= \lim_{z \rightarrow z_1} \frac{i4(2z - z + z_2)}{a^4(z-z_2)^3}$$

$$= \lim_{z \rightarrow z_1} \frac{i4(z + z_2)}{a^4(z-z_2)^3}$$

$$= \frac{-i4z_1}{a^4(z_1-z_2)^3} = \dots = \frac{-i}{a(1-a^2)^{3/2}}$$

→ integral:

$$I = 2\pi i \cdot \frac{(1-i)}{a(1-a^2)^{3/2}}$$

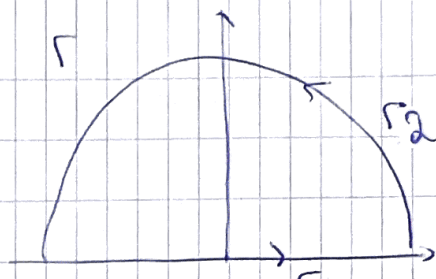
$$= \frac{2\pi}{(1-a^2)^{3/2}}$$

EXTRA BLAD

Jordan uitpassen:

$$\text{Stel } \lim_{z \rightarrow +\infty} f(z) = 0$$

met $\arg z \in [0, \pi]$ zodat $z \in [\Gamma]$



$$\text{Dan } \int_{\Gamma_2} f(z) e^{iaz} dz \rightarrow 0.$$

Bewijs

$$\begin{aligned} & \left| \int_{\Gamma_2} f(z) e^{iaz} dz \right| \\ &= \left| \int_0^\pi f(Re^{i\theta}) e^{iaRe^{i\theta}} iRe^{i\theta} d\theta \right| \\ &= \left| \int_0^\pi f(Re^{i\theta}) e^{iaR(\cos\theta + i\sin\theta)} iRe^{i\theta} d\theta \right| \\ &= \left| \int_0^\pi f(Re^{i\theta}) e^{iaR\cos\theta - aR\sin\theta} iRe^{i\theta} d\theta \right| \\ &= \left| \int_0^\pi f(Re^{i\theta}) e^{iaR\cos\theta} e^{-aR\sin\theta} iRe^{i\theta} d\theta \right| \end{aligned}$$

$$\leq (2R) \max |f(Re^{i\theta})| \int_0^{\pi/2} e^{-aR\sin\theta} d\theta$$

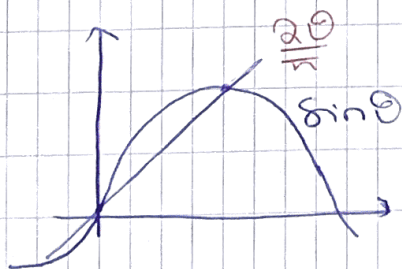
$$\leq (2R) \max |f(Re^{i\theta})| \int_0^{\pi/2} e^{-\frac{2Ra\theta}{\pi}} d\theta$$

$$= (2R) \max |f(Re^{i\theta})| \frac{\pi}{2Ra} (1 - e^{-aR}) \rightarrow 0$$

$$0 \leq \theta \leq \frac{\pi}{2} : \sin\theta \geq \frac{2\theta}{\pi}$$

$$\Rightarrow e^{-aR\sin\theta} \leq e^{-\frac{2aR\theta}{\pi}}$$

□



warmed mit 2.5.13. def 5)

$$\int_{\Gamma_2} \frac{z e^{-iaz}}{z^n + 1} dz \quad \text{hier ist } f(z) = \frac{z}{z^n + 1}$$

$$\rightarrow \left| \int_{\Gamma_2} \frac{z}{z^n + 1} e^{-iaz} dz \right| \quad \text{mit } z = Re^{i\theta} \text{ mit } \theta \in [0, \pi]$$

$$= \left| \int_0^\pi \frac{Re^{i\theta}}{Re^{i\theta} + 1} e^{-i a R e^{i\theta}} i R e^{i\theta} d\theta \right|$$

$$\leq 2R \max \left| \frac{Re^{i\theta}}{Re^{i\theta} + 1} \right| \int_0^{\pi/2} e^{-aR \sin \theta} d\theta$$

$$\leq 2R \max \left| \frac{Re^{i\theta}}{Re^{i\theta} + 1} \right| \int_0^{\pi/2} e^{-\frac{2Ra\theta}{\pi}} d\theta$$

$$\text{Set } -\frac{2Ra\theta}{\pi} = t$$

$$\Rightarrow \frac{d\theta}{d\theta} = \frac{\pi dt}{-2Ra}$$

$$\text{als } \theta \rightarrow \pi/2 \text{ dann } -\frac{2Ra\theta}{\pi} = t \rightarrow -Ra$$

$$\text{als } \theta \rightarrow 0 \text{ dann } t \rightarrow 0$$

$$= 2R \max \left| \frac{Re^{i\theta}}{Re^{i\theta} + 1} \right| \left[\frac{-\pi e^t}{2Ra} \right]_{t=0}^{t=-Ra}$$

$$= 2R \max \left| \frac{Re^{i\theta}}{Re^{i\theta} + 1} \right| \left(\frac{1 - e^{-aR}}{2Ra} \right) \pi$$

$$= \frac{1 - e^{-aR}}{2Ra} \pi$$

$$\rightarrow 0 \text{ als } R \rightarrow +\infty$$

oefening 2.5.13.

$$1) \int_{-\infty}^{+\infty} \frac{1-x}{1-x^5} dx$$

we bekijken $\int_{\Gamma} \frac{1-z}{1-z^5} dz$ met $f(z) = \frac{1-z}{1-z^5}$

→ singuliere punten van $f(z)$:
nulpunten wilt noemen:

$$\begin{aligned} 1 - z^5 &= 0 \\ \Leftrightarrow z^5 &= 1 \\ \Leftrightarrow z^5 &= e^{2\pi i k} \end{aligned}$$

$$\Leftrightarrow z = \exp\left(\frac{2\pi i k}{5}\right), k \in \{0, 1, 2, 3, 4\},$$

enkel $\exp\left(\frac{2\pi i}{5}\right)$ en $\exp\left(\frac{4\pi i}{5}\right) \in [\Gamma]$

→ residu's bepalen:

$$z = \exp(2\pi i/5):$$

$$\text{res}_{z=z_1} f(z) = \lim_{z \rightarrow z_1} \frac{(z-z_1)(1-z)}{1-z^5}$$

$$= \lim_{z \rightarrow z_1} \frac{(z - \exp(2\pi i/5))(1-z)}{1-z^5}$$

$$= \lim_{z \rightarrow z_1} (1-z) \lim_{z \rightarrow z_1} \frac{(z - \exp(2\pi i/5))}{1-z^5} \stackrel{0/0}{=}$$

$$\stackrel{H}{=} [1 - \exp(2\pi i/5)] \lim_{z \rightarrow z_1} \frac{1}{-5z^4}$$

$$= [1 - \exp(2\pi i/5)] \cdot \frac{1}{-5} \exp(-i8\pi/5)$$

$$= \frac{\exp(4\pi i/5) - \exp(2\pi i/5)}{5}$$

analogo in

$$\text{res}_{z=\exp(4\pi i/5)} f(z) = \frac{\exp(8\pi i/5) - \exp(4\pi i/5)}{5}$$

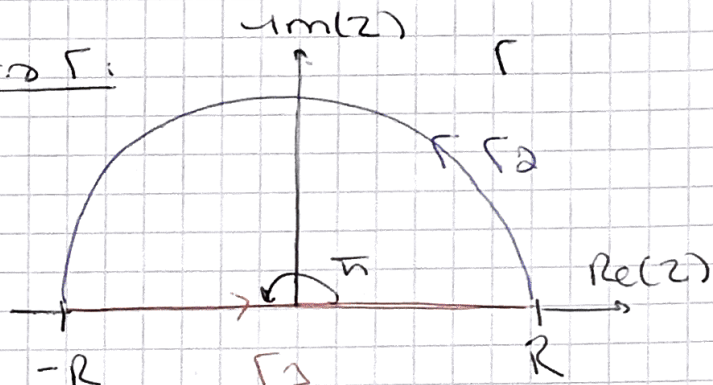
→ via residu'srelling vinden we dat

$$\int_{\Gamma} \frac{1-z}{1-z^5} dz = 2\pi i \left(\frac{\exp(8\pi i/5) - \exp(2\pi i/5)}{5} \right)$$

$$= \frac{2\pi i}{5} (\exp(-2\pi i/5) - \exp(2\pi i/5))$$

$$= \frac{2\pi i}{5} \cdot 2i \sin\left(-\frac{2\pi}{5}\right)$$

$$= \frac{4\pi}{5} \sin\left(\frac{2\pi}{5}\right)$$



enkelvoudige
polen!!

Er gelte $\lim_{z \rightarrow \infty} f(z) = 0$

$$\int_{\gamma_+} f(z) dz = \int_{\gamma_+} f(z) dz + \int_{\gamma_-} f(z) dz$$

mit

$$\underbrace{\int_{\gamma_+} f(z) dz}_{\text{Reile-as}} = \int_{-R}^R \frac{1-x}{1-x^5} dx \xrightarrow{R \rightarrow +\infty} \int_{-\infty}^{+\infty} \frac{1-x}{1-x^5} dx$$

und

$$\left| \int_{\gamma_-} f(z) dz \right| = \left| \int_0^{2\pi} \frac{1-Re^{it}}{1-R^5 e^{i5t}} i R e^{it} dt \right|$$

$z = Re^{it} \quad (0 \leq t \leq 2\pi)$

$$\leq \frac{(R+1)R}{R^5-1} \rightarrow 0 \text{ als } R \rightarrow +\infty$$

mit n immer verkleinern!!

↳ Consequenz: (Zodre $R > 1$)

$$\int_{\gamma_+} f(z) dz + \int_{\gamma_-} f(z) dz = \int_{\gamma_+} f(z) dz$$

$$\Rightarrow \int_{-R}^R f(x) dx + 0 = \frac{4\pi}{5} \sin\left(\frac{2\pi}{5}\right)$$

$$\xrightarrow{R \rightarrow +\infty} \int_{-\infty}^{+\infty} \frac{1-x}{1-x^5} dx = \frac{4\pi}{5} \sin\left(\frac{2\pi}{5}\right)$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{1-x}{1-x^5} dx = \frac{4\pi}{5} \sin\left(\frac{2\pi}{5}\right)$$

↳ want $\int_{-\infty}^{+\infty} f(x) dx$ ist integrierbar.

4) a) $\int_0^{+\infty} x^{-n} dx = \int_0^{+\infty} \frac{u^{-n}}{\frac{1}{n}} u^{\frac{1}{n}-1} du$
 $= \frac{1}{n} \Gamma\left(\frac{1}{n}\right)$ (zie analyse 1)

b) $\int_0^{+\infty} \cos(x^n) dx = \operatorname{Re} \int_0^{+\infty} e^{ix^n} dx$

→ singuliere punten kapelen:

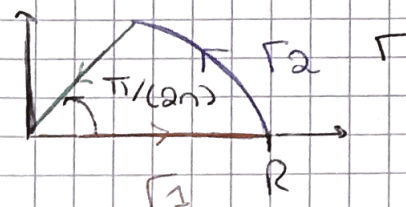
Er zijn er geen!

zodat we via de kloofomkering
 vinden dat

$$\oint_{\Gamma} f(z) dz = 0$$

Nu bekijken we de delen wat contour afzonderlijk

rechter:



Γ1: $\int_{\Gamma_1} e^{iz^n} dz = \int_0^R e^{ix^n} dx \rightarrow$ gevraagde!

Γ2: $\left| \int_{\Gamma_2} e^{iz^n} dz \right| = \left| \int_0^{\pi/(2n)} e^{-R^n \sin^n t} R^{n-1} e^{iR^n \cos^n t} R i e^{it} dt \right|$

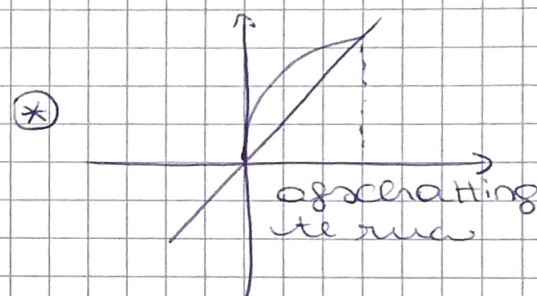
$\leq \int_0^{\pi/(2n)} e^{-R^n \sin^n t} R^n dt$

$\leq \int_0^{\pi/(2n)} e^{-\frac{R^n}{\pi} \sin t} R^n dt$

$= \left[\frac{R \exp(-\frac{R^n \sin t}{\pi})}{R^n \sin t / \pi} \right]_{t=0}^{t=\frac{\pi}{2n}}$

$= \frac{\pi R^{1-n}}{2n} (1 - e^{-R^n})$

$\rightarrow 0 \quad (n \geq 2)$



$\sin x \geq \frac{2}{\pi} x \quad \text{als}$
 $0 \leq x \leq \frac{\pi}{2}$

(ongelijkheid van Jordan)

Γ3: $\int_{\Gamma_3} e^{iz^n} dz = \int_0^R e^{-x^n} e^{i\pi/(2n)} dx$
 $= - \int_0^R e^{-x^n} e^{i\pi/(2n)} dx$
 $\xrightarrow{\text{deels a)}} - \frac{e^{i\pi/(2n)}}{n} \Gamma\left(\frac{1}{n}\right)$

Conclusie:

$\oint_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \int_{\Gamma_3} f(z) dz = 0$

$\Rightarrow \lim_{R \rightarrow +\infty} \int_{\Gamma_R} f(z) dz$ herhaalt

$$\text{en } \lim_{R \rightarrow +\infty} \int_{\Gamma_R} f(z) dz = -\frac{e^{\frac{j\sqrt{\pi}}{2n}} \Gamma(\frac{1}{2n})}{n}$$

zodat

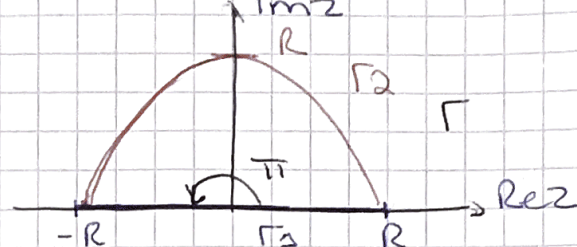
$$\int_0^{\infty} e^{ix^m} dx = \frac{e^{\frac{j\sqrt{\pi}}{2n}} \Gamma(\frac{1}{2n})}{n}$$

$\rightarrow$
reële deel
nemen

$$\int_0^{\infty} \cos(x^m) dx = \frac{\cos(\frac{\sqrt{\pi}}{2n}) \Gamma(\frac{1}{2n})}{n}$$

oefening 2.5.13. af 5)

$$\int_{-\infty}^{+\infty} \frac{x \sin x}{x^4 + 4} dx \quad (a > 0)$$



we bekijken $\int_{-\infty}^{+\infty} \frac{z \sin z}{z^4 + 4} dz$.

- singuliere punten uropalen

→ nulpunten van $z^4 + 4$:

$$z^4 + 4 = 0 \Rightarrow z^4 = -4 \Rightarrow z^2 = -2i \vee z^2 = 2i$$

ook $z = x + iy$ ($x, y \in \mathbb{R}$):

$$(x + iy)^2 = -2i \vee (x + iy)^2 = 2i$$

$$\Rightarrow \begin{cases} x^2 - y^2 = 0 \\ 2xy = -2 \end{cases} \vee \begin{cases} x^2 - y^2 = 0 \\ 2xy = 2 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 = y^2 \\ xy = -1 \end{cases} \vee \begin{cases} x^2 = y^2 \\ xy = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 = (\frac{1}{x})^2 \\ y = -1/x \end{cases} \vee \begin{cases} x^2 = (\frac{1}{x})^2 \\ y = 1/x \end{cases}$$

$$\Rightarrow \begin{cases} x = \pm 1 \\ y = \mp 1 \end{cases} \vee \begin{cases} x = \pm 1 \\ y = \pm 1 \end{cases}$$

$$\Rightarrow z = \pm(-1+i) \vee z = \pm(1+i)$$

$$z = 1-i \wedge z = -1-i \notin [\Gamma]$$

ook we hebben $z = -1+i$ en $z = 1+i$ als singuliere punten $\Rightarrow$ beide polen van orde 1

- residu uropalen:

$$\text{voor } z = -1+i: \text{res}_{z=-1+i} f(z) = \lim_{z \rightarrow -1+i} \frac{(z - (-1+i)) z \sin(az)}{(z^2 - 2i)(z - (-1+i))(z + (-1+i))}$$

$$= \lim_{z \rightarrow -1+i} \frac{z \sin(az)}{(z^2 - 2i)(z + i - 1)} = \frac{(-1+i) \sin(a(-1+i))}{(-2i - 2i)(-1+i+i-1)}$$

$$= \frac{\sin(a(-1+i))}{-8i} \cdot \frac{i}{i} = \frac{\sin(a(-1+i))i}{8}$$

$$\text{voor } z = 1+i: \text{res}_{z=1+i} f(z) = \lim_{z \rightarrow 1+i} \frac{z \sin(az)}{(z^2 + 2i)(z + (1+i))}$$

$$= \frac{(1+i) \sin(a(1+i))}{(4i)(2+2i)} = \frac{\sin(a(1+i))}{8i} \cdot \frac{i}{i}$$

$$= \frac{-\sin(a(1+i))i}{8}$$

- via residuerekening:

$$\int_{\Gamma} f(z) dz = 2\pi i \left[\frac{\sin(a(-1+i))}{b} - \frac{\sin(a(1+i))}{b} \right]$$

$$= \frac{\pi}{b} [\sin(a(1+i)) - \sin(a(-1+i))]$$

$$= \frac{\pi}{b} \left[\frac{e^{a(1+i)i} - e^{-a(1+i)i}}{2i} - \frac{e^{a(-1+i)i} - e^{-a(-1+i)i}}{2i} \right]$$

$$= \frac{\pi}{b} \left[\frac{e^{a(i-1)} - e^{-a(i-1)}}{2i} - \frac{e^{a(-i-1)} - e^{-a(-i-1)}}{2i} \right]$$

$$= \frac{\pi}{b} \left[\frac{e^{-a} e^{ia} - e^{-ai} e^a - e^{-a} e^{-ai} + e^{ai} e^a}{2i} \right]$$

$$= \frac{\pi}{b} \left[e^{-a} \left(\frac{e^{ia} - e^{-ai}}{2i} \right) + e^a \left(\frac{e^{ai} - e^{-a}}{2i} \right) \right]$$

$$= \frac{\pi}{b} (e^{-a} \sin a + e^a \sin a) = \frac{\pi}{b} (e^{-a} + e^a) \sin a$$

→ $\int_{\Gamma_2} f(z) dz$ ergibt dat $\sin(az)$ het imaginair deel is van e^{iaz} .

$$\left| \int_{\Gamma_2} \frac{ze^{iaz}}{z^b + b} dz \right| \stackrel{z = Re^{it}}{0 \leq t \leq \pi} = \left| \int_0^\pi \frac{Re^{it} e^{iade^{it}}}{R^b e^{ibt} + b} Rie^{it} dt \right|$$

$$= \left| \int_0^\pi \frac{Re^{it} (i a (e^{i a \cos t} + i \sin t))}{R^b e^{ibt} + b} Rie^{it} dt \right|$$

$$\leq \left| \int_0^\pi \frac{R^2 e^{it} - R a \cos t i e^{-it}}{R^b e^{ibt} + b} dt \right|$$

$$\leq \left| \int_0^\pi \frac{R^2 e^{-R a \cos t}}{R^b e^{ibt} + b} dt \right| \rightarrow 0 \text{ als } R \rightarrow \infty$$

die extra
beoord

$$\rightarrow \int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz$$

$$\frac{\pi}{b} (e^{-a} + e^a) \sin a = \int_{\Gamma_1} f(z) dz + 0$$

$$\frac{\pi}{b} (e^{-a} + e^a) \sin a = \int_{-R}^R \frac{x \sin(ax)}{x^b + b} dx$$

$$\stackrel{R \rightarrow \infty}{\rightarrow} \frac{\pi}{b} (e^{-a} + e^a) \sin a = \int_{-\infty}^{+\infty} \frac{x \sin(ax)}{x^b + b} dx$$

oef 6)

$$\int_{-\infty}^{+\infty} \frac{\sin(\pi x)}{x(x-1)(x-2)} dx$$

$f(x)$

De nulpunten van de noemer van $f(x)$ zorgen voor ophefbare discontinuïteiten

$\Rightarrow$ We mogen dit doorintegreren

We bekijken $\int_{\Gamma} f(z) dz$ en we geldt dat

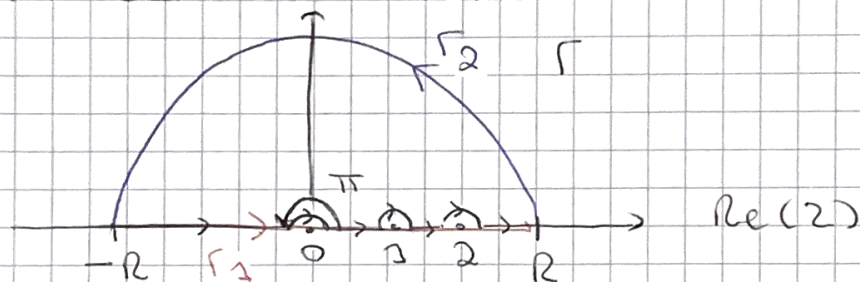
$$\int_{\Gamma} f(z) dz = \lim \left(\int_{\Gamma} \frac{e^{i\pi z}}{z(z-1)(z-2)} dz \right)$$

$\rightarrow$ Singuliere punten van $\frac{e^{i\pi z}}{z(z-1)(z-2)}$ in Γ :

$$z(z-1)(z-2) = 0$$

$\Rightarrow z=0 \vee z=1 \vee z=2 \rightarrow$ allemaal mult-
 $\rightarrow$ enkelvoudige polen

schets van $\Gamma: \text{Im}(z)$



$\rightarrow$ residu's berekenen:

$$\begin{aligned} \text{res}_{z=0} \left(\frac{e^{i\pi z}}{z(z-1)(z-2)} \right) &= \lim_{z \rightarrow 0} \frac{e^{i\pi z}}{(z-1)(z-2)} \\ &= \frac{e^0}{-2 \cdot (-1)} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{res}_{z=1} \left(\frac{e^{i\pi z}}{z(z-1)(z-2)} \right) &= \lim_{z \rightarrow 1} \frac{e^{i\pi z}}{z(z-2)} \\ &= \frac{e^{i\pi}}{1(1-2)} = \frac{-1}{-1} = 1 \end{aligned}$$

$$\begin{aligned} \text{res}_{z=2} \left(\frac{e^{i\pi z}}{z(z-1)(z-2)} \right) &= \lim_{z \rightarrow 2} \frac{e^{i\pi z}}{z(z-1)} \\ &= \frac{e^{2\pi i}}{2(2-1)} = \frac{1}{2} \end{aligned}$$

$\rightarrow$ via lemma 2.5.10 vinden we dan

$$\int_{\Gamma} \frac{e^{i\pi z}}{z(z-1)(z-2)} dz = \pi i \left(\frac{1}{2} + 1 + \frac{1}{2} \right) = 2\pi i$$

verander in

$$\int_{\Gamma} \frac{e^{i\pi z}}{z(z-1)(z-2)} dz = \int_{\Gamma_1} \frac{e^{i\pi z}}{z(z-1)(z-2)} dz + \int_{\Gamma_2} \frac{e^{i\pi z}}{z(z-1)(z-2)} dz$$

we maken deze niet
opspitsen wegens de

ophefbaarheid van de Polen van $\ln\left(\frac{e^{i\pi z}}{z(z-1)(z-2)}\right)$

$$\Gamma_2: \left| \int_{\Gamma_2} \frac{1}{z(z-1)(z-2)} e^{i\pi z} dz \right|$$

$\hookrightarrow$ stel $g(z)$, ergeldt dat
 $g(z) \rightarrow 0$ als $|z| \rightarrow +\infty$

$$= \left| \int_{\Gamma_2} g(z) e^{i\pi z} dz \right|$$

$$\left(z = Re^{i\theta} (\theta \in [0, \pi]) \right)$$

$$= \left| \int_0^\pi g(Re^{i\theta}) e^{i\pi Re^{i\theta}} R i e^{i\theta} d\theta \right|$$

$$\leq (2R) \max |g(Re^{i\theta})| \int_0^{\pi/2} e^{-\pi R \sin \theta} d\theta$$

$$\leq 2R \max |g(Re^{i\theta})| \int_0^{\pi/2} e^{-2R\theta} d\theta$$

$$0 \leq \theta \leq \frac{\pi}{2} \Rightarrow \sin \theta \geq \frac{2\theta}{\pi}$$

$$\Rightarrow e^{-\pi R \sin \theta} \leq e^{-\frac{2R\theta}{\pi}}$$

$$= 2R \max |g(Re^{i\theta})| \left(\frac{-1}{2R} \right) \left[e^{-2R\theta} \right]_0^{\pi/2}$$

$$= 2R \max |g(Re^{i\theta})| \left(\frac{1 - e^{-2R\pi/2}}{2R} \right)$$

$$\rightarrow 0 \text{ als } R \rightarrow +\infty$$

zodat

$$\int_{\Gamma_1} \frac{e^{i\pi z}}{z(z-1)(z-2)} dz = 2\pi i$$

$$\int_{-R}^R \frac{e^{i\pi x}}{x(x-1)(x-2)} dx = 2\pi i$$

$$\rightarrow \int_{-\infty}^{+\infty} \frac{e^{i\pi x}}{x(x-1)(x-2)} dx = 2\pi i$$

echter enkel het imaginair deel is
gevraagd zodat

$$\int_{-\infty}^{+\infty} f(x) dx = 2\pi$$

Def 9.1

$$\int_0^{+\infty} \frac{x e^x}{e^{2x} - 1} dx$$

we bekijken $\oint_{\Gamma} \frac{f(z)}{e^{2z} - 1} dz$

→ singuliere punten van $f(z)$:

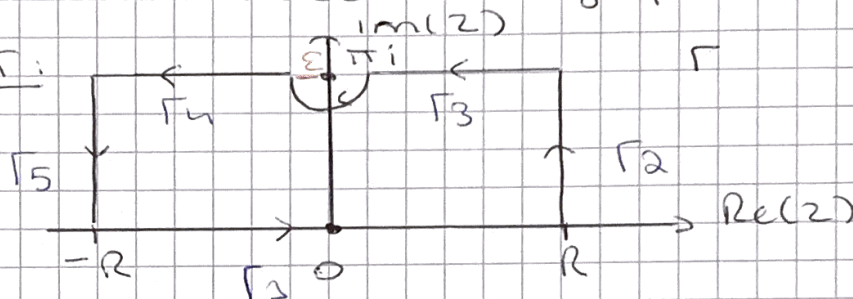
$$e^{2z} - 1 = 0 \Leftrightarrow e^{2z} = 1 = e^{i2\pi k}$$

$$\Rightarrow z = \pi k i \text{ stel } k=1 \text{ en } k=0$$

mult $f(z) = 0 \rightarrow$ ophefbare singulariteit
 $z=0$

mult $f(z) = -1 \rightarrow$ enkelvoudige pool
 $z=1$

schets van Γ :



→ residu's bepalen:

$$\text{res}_{z=\pi i} f(z) = \lim_{z \rightarrow \pi i} \frac{(z - \pi i) z e^z}{e^{2z} - 1} = \lim_{z \rightarrow \pi i} \frac{(z^2 - 2\pi i) e^z}{e^{2z} - 1}$$

$$\stackrel{L}{=} \lim_{z \rightarrow \pi i} \frac{(2z - \pi i) e^z + (z^2 - 2\pi i) e^z}{2e^{2z}}$$

$$= \lim_{z \rightarrow \pi i} \frac{2z - \pi i + z^2 - 2\pi i}{2e^{2z}}$$

$$= \frac{2\pi i - \pi i + (\pi i)^2 - 2(\pi i)^2}{2e^{2\pi i}}$$

$$= \frac{\pi i}{2(\cos \pi + i \sin \pi)} = -\frac{\pi i}{2}$$

→ via lemma 2.5.10:

$$\oint_{\Gamma} f(z) dz = -\pi i \cdot \frac{\pi i}{2} = \frac{\pi^2}{2}$$

$$\boxed{\Gamma_2} \quad \left| \oint_{\Gamma_2} f(z) dz \right| = \left| \int_0^{\pi} \frac{i(R+iy)e^{R+iy}}{e^{2R+2iy} - 1} dy \right|$$

$z = R+iy$
 $y \in [0, \pi]$

stel $g(y)$

als $R \rightarrow +\infty$ dan $|g(y)| \rightarrow 0$

$$\leq \max_{y \in [0, \pi]} |g(y)| \int_0^{\pi} e^{-y} dy$$

$$\begin{aligned}
 &\leq \max_{y \in [0, \pi]} |g(y)| \left| \int_0^{\pi} e^{iy} dy \right| \\
 &= \max_{y \in [0, \pi]} |g(y)| \left| \frac{1}{i} [e^{iy}]_0^{\pi} \right| \\
 &= \max_{y \in [0, \pi]} |g(y)| \cdot \underbrace{(1 - e^{i\pi})}_{-1} \\
 &\rightarrow 0 \text{ als } R \rightarrow +\infty
 \end{aligned}$$

$$\boxed{\Gamma_5}: \left| \int_{\Gamma_5} \frac{ze^z}{e^{2z}-1} dz \right|_{z=-R+iy} = \left| \int_0^{\pi} \frac{(-R+iy)e^{-R+iy}}{e^{-R+iy}-1} dy \right|$$

$$\begin{aligned}
 &\lim_{R \rightarrow +\infty} \frac{(-R+iy)e^{-R}}{e^{-R}(e^{iy}-1)} = \lim_{R \rightarrow +\infty} \frac{-R+iy}{-e^R + e^{iy}} \\
 &= \frac{-\infty}{-\infty} \stackrel{\text{L'H}}{=} \lim_{R \rightarrow +\infty} \frac{+1}{+e^R} = 0
 \end{aligned}$$

$$\leq \max_{y \in [0, \pi]} \left| \frac{(-R+iy)e^{-R}}{e^{-R}e^{iy}-1} \right| \left| \int_0^{\pi} e^{iy} dy \right|$$

$$\rightarrow 0 \text{ als } R \rightarrow +\infty$$

$$\boxed{\Gamma_3}: \int_{\Gamma_3} f(z) dz = \lim_{\varepsilon \rightarrow 0+} \int_{\varepsilon}^R \frac{(x+\pi i)e^{x+\pi i}}{e^{2x+2\pi i}-1} dx$$

s. uel $z = x + \pi i$
 $x \in [\varepsilon, R]$

$$= - \lim_{\varepsilon \rightarrow 0+} \int_{\varepsilon}^R \frac{(x+\pi i)e^x e^{\pi i}}{e^{2x} \cdot e^{2\pi i} - 1} dx$$

$$= - \lim_{\varepsilon \rightarrow 0+} \int_{\varepsilon}^R \frac{\pi i e^{\pi i} e^x + x e^{\pi i} e^x}{e^{2x} - 1} dx$$

$$= - \lim_{\varepsilon \rightarrow 0+} \left[\int_{\varepsilon}^R \frac{\pi i e^{\pi i} e^x}{e^{2x} - 1} dx + \int_{\varepsilon}^R \frac{x e^{\pi i} e^x}{e^{2x} - 1} dx \right]$$

orientierung!

$$= \underbrace{(+)\frac{1}{2} e^{\pi i} \int_{\Gamma_3} f(z) dz}_{\text{orientierung}} - \lim_{\varepsilon \rightarrow 0+} \int_{\varepsilon}^R \frac{\pi i e^{\pi i} e^x}{e^{2x} - 1} dx$$

$$\boxed{\Gamma_4}: \int_{\Gamma_4} f(z) dz = \lim_{\varepsilon \rightarrow 0+} \int_{\varepsilon}^{-R} \frac{(x+\pi i)e^{x+\pi i}}{e^{2x+2\pi i}-1} dx$$

$$= \lim_{\varepsilon \rightarrow 0+} \left[\int_{\varepsilon}^{-R} \frac{x e^x e^{\pi i}}{e^{2x} - 1} dx + \int_{\varepsilon}^{-R} \frac{\pi i e^{\pi i} e^x}{e^{2x} - 1} dx \right]$$

$$= \underbrace{(+)\frac{1}{2} e^{\pi i} \int_{\Gamma_4} f(z) dz}_{\text{orientierung}} + \lim_{\varepsilon \rightarrow 0+} \int_{\varepsilon}^{-R} \frac{\pi i e^{\pi i} e^x}{e^{2x} - 1} dx$$

addat

$$\int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \int_{\Gamma_3} f(z) dz + \int_{\Gamma_4} f(z) dz + \int_{\Gamma_5} f(z) dz = \frac{\pi^2}{2}$$

$$\Rightarrow 2 \int_{\Gamma_1} f(z) dz + \lim_{\varepsilon \rightarrow 0+} \left[\int_{\varepsilon}^{-R} \frac{-\pi i e^x}{e^{2x}-1} dx + \int_{\varepsilon}^R \frac{\pi i e^x}{e^{2x}-1} dx \right] + 0 + 0 = \frac{\pi^2}{2}$$

= 0 wegens symmetrie!!

$$\Rightarrow 2 \int_{\Gamma_1} f(z) dz = \frac{\pi^2}{2}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{x e^x}{e^{2x}-1} dx = \frac{\pi^2}{4}$$

$$\Rightarrow \int_0^{+\infty} \frac{x e^x}{e^{2x}-1} dx = \frac{\pi^2}{8} \quad \checkmark$$

af 10) $\int_0^{+\infty} e^{-x^2} \sin(x^2) dx$

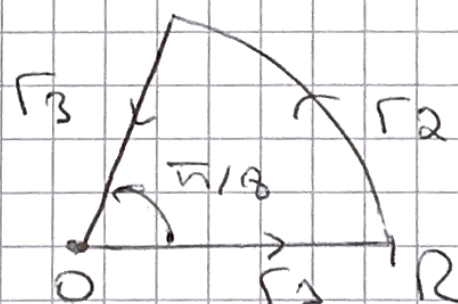
we bekijken $\int_{\Gamma} e^{-z^2} e^{iz^2} dz$

addat $\lim (\int_{\Gamma} e^{-z^2} e^{iz^2} dz) = \int_{\Gamma} e^{-z^2} \sin(z^2) dz$

$$\hookrightarrow \int_{\Gamma} e^{-z^2} e^{iz^2} dz = \int_{\Gamma} e^{z^2(i-1)} dz$$

→ er zijn geen singuliere punten

ocht van Γ :



→ we passen de hoofdstelling toe:

$$\int_{\Gamma} f(z) dz = 0$$

$$\boxed{\Gamma_2} \quad \left| \int_{\Gamma_2} f(z) dz \right| = \left| \int_0^{\pi/8} i R e^{i\theta} e^{R^2 e^{2i\theta}} (i-1) d\theta \right|$$

$\hookrightarrow |e^{i\theta}| = 1$

$$\leq R \int_0^{\pi/8} e^{R^2 (\cos(2\theta) + i \sin(2\theta)) (i-1)} d\theta$$

$$= R \int_0^{\pi/8} e^{-R^2 (\cos(2\theta) + \sin(2\theta))} d\theta$$

$$\leq R \int_0^{\pi/8} e^{-R^2} d\theta = \frac{R}{e^{R^2}} \frac{\pi}{8} \rightarrow 0 \text{ als } R \rightarrow \infty$$

$$(73) \quad \text{Sei } z = t e^{i\frac{\pi}{8}} \Rightarrow dz = e^{i\frac{\pi}{8}} dt$$

$$\Rightarrow \int_{\gamma_3} f(z) dz$$

$$= -e^{\frac{i\pi}{8}} \int_0^R e^{t^2} e^{\frac{i\pi}{8}} (i-1) dt$$

$$= -e^{i\pi/8} \int_0^R e^{t^2} (ie^{\frac{i\pi}{8}} - e^{\frac{i\pi}{8}}) dt$$

$$= -\exp(i\frac{\pi}{8}) \int_0^R \exp(t^2 (i(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}) - \cos\frac{\pi}{8} - i\sin\frac{\pi}{8})) dt$$

$$= -\exp(i\frac{\pi}{8}) \int_0^R \exp(t^2 (\frac{\sqrt{2}}{2} i(1+i) - \frac{\sqrt{2}}{2} (1+i))) dt$$

$$= -\exp(i\frac{\pi}{8}) \int_0^R \exp(t^2 (1+i) (\frac{\sqrt{2}}{2} (i-1))) dt$$

$$= -\exp(i\frac{\pi}{8}) \int_0^R \exp(t^2 \sqrt{2}) dt$$

$$= \frac{-\exp(i\pi/8)}{\sqrt{2}} \int_0^{R/\sqrt{2}} \exp(u^2) du$$

$$= \frac{-\exp(i\pi/8) \sqrt{\pi}}{2\sqrt{2}}$$

200101

$$\int_{\gamma_3} g(z) dz = \frac{\exp(i\pi/8) \sqrt{\pi}}{2\sqrt{2}}$$

$$= \frac{(\cos(\pi/8) + i\sin(\pi/8)) \sqrt{\pi}}{2\sqrt{2}}$$

$$\Rightarrow \int_0^{+\infty} e^{-x^2} \sin(x^2) dx = \frac{\sin(\pi/8) \sqrt{\pi}}{2\sqrt{2}} \checkmark$$

~~$$\Rightarrow \int_0^{+\infty} e^{-x^2} \sin(x^2) dx = \frac{\sin(\pi/8) \sqrt{\pi}}{2\sqrt{2}}$$~~

EXTRA OEFENINGEN (online gehaald)

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x+1)}$$

↳ $x = -1$ niet ophefbaar!

we bekijken $\oint_{\Gamma} \frac{dz}{(z^2+1)(z+1)} = f(z)$

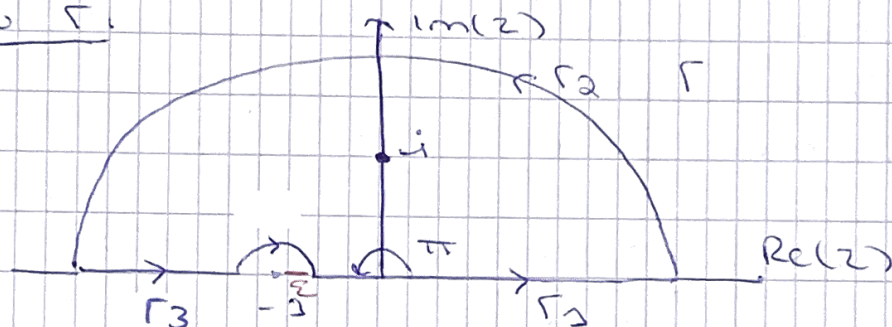
→ singuliere punten:

$$(z^2+1)(z+1) = 0 \Leftrightarrow z^2+1 = 0 \vee z = -1$$

$$\Leftrightarrow z = \pm i \vee z = -1$$

enkel $z_1 = i$ en $z = -1 \in [\Gamma]$.

ochers w Γ .



mult $f(z)$ at $z = z_1 = i$ = mult $f(z)$ at $z = z_2 = -1$ = -1 → enkelvoudige Polen!

→ residu's bepalen:

$$\text{res}_{z=i} f(z) = \lim_{z \rightarrow i} \frac{1}{(z+i)(z+1)} = \frac{1}{2i(i+1)}$$

$$= \frac{1}{-2+2i} = \frac{1}{2} \cdot \frac{1}{-1+i} \cdot \frac{(-1-i)}{(-1-i)}$$

$$= \frac{1}{2} \cdot \frac{(-1-i)}{1+1} = \frac{-1-i}{4}$$

$$\text{res}_{z=-1} f(z) = \lim_{z \rightarrow -1} \frac{1}{z^2+1} = \frac{1}{2}$$

→ via 'residuëering' en lemma 2.5.10'

worden we

$$\oint_{\Gamma} f(z) dz = i\pi \left(\frac{1}{2} \right) + \frac{-i2\pi(1+i)}{4}$$

$$= \frac{\pi i}{2} - \frac{i\pi}{2} + \frac{\pi}{2} = \frac{\pi}{2}$$

hat gevraagd: Teil I

$$\Rightarrow \int_{\Gamma} f(z) dz = \lim_{\varepsilon \rightarrow 0+} \left(\int_{\Gamma_3} f(z) dz + \int_{\Gamma_1} f(z) dz \right) + \int_{\Gamma_2} f(z) dz :$$

$$\Gamma_2: \left| \int_{\Gamma_2} \frac{1}{(z^2+1)(z-1)} dz \right|$$

$$(\lim_{|z| \rightarrow \infty} f(z) = 0)$$

$$= \left| \int_0^\pi f(Re^{i\theta}) e^{i\theta} iR d\theta \right|$$

$$\leq \max |f(Re^{i\theta})| \int_0^\pi e^{i\theta} iR d\theta$$

$$\leq R \max |g(Re^{i\theta})| \int_0^\pi e^{i\theta} d\theta$$

$$= |R \max |g(Re^{i\theta})| \left[\frac{e^{i\theta}}{i} \right]_0^\pi|$$

$$= |R \max |g(Re^{i\theta})| \cdot (1 - e^{i\pi})|$$

$$\rightarrow 0 \text{ als } R \rightarrow +\infty$$

$$\rightarrow 0 \text{ als } R \rightarrow +\infty$$

zusatz

$$\boxed{I = \frac{\sqrt{\pi}}{2}}$$

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, \quad 0 < a < 1$$

we beschauen $\oint \frac{e^{az}}{1+e^z} dz = f(z)$

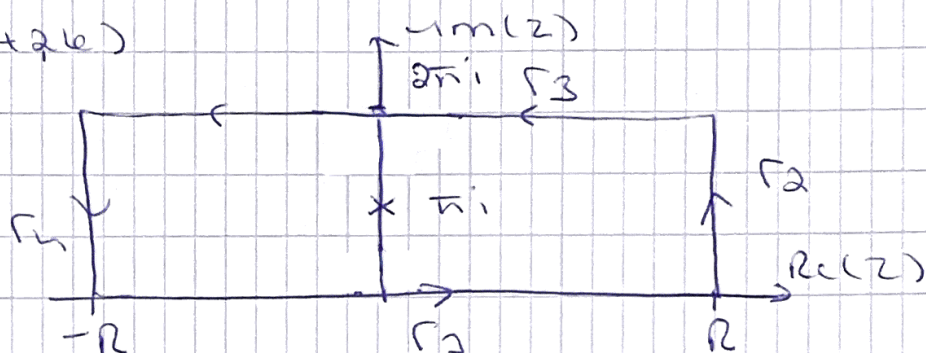
→ singuläre punkte von $f(z)$:

$$1+e^z = 0 \Leftrightarrow -1 = e^z$$

$$\Leftrightarrow e^{\pi i + 2k\pi i} = e^z \quad k \in \mathbb{Z}$$

$$\Leftrightarrow z = \pi i (1+2k)$$

schets w. Γ :



→ residu's berechnen:

$$\operatorname{res}_{z=\pi i} f(z) = \lim_{z \rightarrow \pi i} \left((z - \pi i) \frac{e^{az}}{1+e^z} \right)$$

$$\stackrel{L'H}{=} \lim_{z \rightarrow \pi i} \left(\frac{-e^{az} + (z - \pi i) a e^{az}}{e^z} \right)$$

$$= \lim_{z \rightarrow \pi i} \left(\frac{e^{az} (1 + az - \pi i a)}{e^z} \right)$$

$$= \frac{e^{a\pi i} (1 + a\pi i - \pi i a)}{-1}$$

$$= -e^{a\pi i}$$

→ via residu-mulling:

$$\oint_{\Gamma} f(z) dz = -2\pi i e^{a\pi i}$$

$$= \underbrace{\int_{\Gamma_1} f(z) dz}_{\text{ger}} + \int_{\Gamma_2} f(z) dz + \int_{\Gamma_3} f(z) dz + \int_{\Gamma_4} f(z) dz$$

$$\Gamma_2: \left| \int_{\Gamma_2} f(z) dz \right| = \left| \int_0^{2\pi} i \frac{e^{aR} e^{iay}}{1+e^{R+iy}} dy \right|$$

set $z = R+iy$
 $y \in [0, 2\pi]$

$$\leq \int_0^{2\pi} \frac{|e^{aR} e^{iay}|}{|1+e^{R+iy}|} dy = \int_0^{2\pi} \frac{e^{aR} |e^{iay}|}{|e^{iy} + e^R|} dy$$

$$\leq \int_0^{2\pi} \frac{|e^{aR} e^{a-1}|}{|e^R + e^{-iy}|} dy \leq \int_0^{2\pi} \frac{|e^{aR} e^{a-1}|}{|e^R - 1|} dy$$

$$= \int_0^{2\pi} \frac{|e^{a-1} e^{(a-1)R}|}{|1 - e^{-R}|} dy \rightarrow 0$$

analog

$$\int_{\Gamma} f(z) dz \rightarrow 0 \quad \text{mit} \quad |1 + e^{-R} e^{iy}| > 1 - e^{-R}$$

$$\Gamma_3: \int_{\Gamma_3} f(z) dz \xrightarrow{z=x+2\pi i} \int_{-R}^{-R} \frac{e^{ia2\pi} e^{ax}}{1+e^x} dx$$

$$= e^{ia2\pi} \int_{-R}^{-R} \frac{e^{ax}}{1+e^x} dx = -e^{ia2\pi} \int_{\Gamma_3} \frac{e^{ax}}{1+e^x} dz$$

zusatz

$$\int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + e^{ia2\pi} \int_{\Gamma_3} f(z) dz$$

$$= (1 + e^{ia2\pi}) \int_{\Gamma_1} f(z) dz$$

$$= -e^{ia2\pi} 2\pi i$$

$$\Rightarrow \int_{\Gamma_1} f(z) dz = \frac{-2\pi i e^{ia2\pi}}{1 - e^{ia2\pi}}$$

$$= \frac{-2\pi i (e^{ia2\pi})}{e^{ia2\pi} (e^{-ia2\pi} - e^{ia2\pi})} = \frac{-\pi}{\sin(-ia2\pi)}$$

$$= \frac{\pi}{\sin(a2\pi)} \quad (0 < a < 1)$$

$$I = \int_0^{\infty} \frac{\cos(\lambda x)}{x^2 + a^2} dx$$

$$= \int_0^{\infty} \frac{e^{-i\lambda x}}{x^2 + a^2} dx - i \int_0^{\infty} \frac{\sin(\lambda x)}{x^2 + a^2} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{x^2 + a^2} dx - \underbrace{\frac{i}{2} \int_{-\infty}^{\infty} \frac{\sin(\lambda x)}{x^2 + a^2} dx}_0$$

wegen Symmetrie

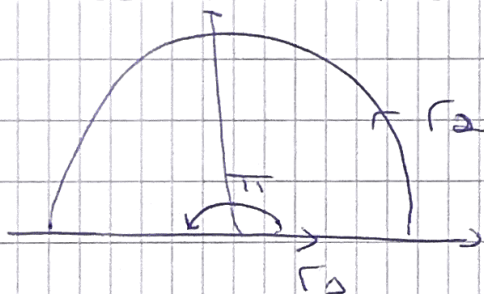
$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{x^2 + a^2} dx$$

→ singuliere punkten:

$$z^2 + a^2 = 0 \Leftrightarrow z = \pm ia$$

$$\text{wobei } z = ia \in (\Gamma)$$

wähle Γ :



$$\rightarrow I = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{e^{-i\lambda z}}{z^2 + a^2} dz$$

$$= 2\pi i \operatorname{res}_{z=ia} f(z) = \frac{2\pi i e^{-\lambda a}}{2 \cdot 2ia} = \frac{\pi e^{-a\lambda}}{2a}$$

$$\int_{-\infty}^{+\infty} \frac{\sin(3x)}{x(x^2+1)} dx$$

$f(z)$

"

we bekijken $\int_{\Gamma} \frac{e^{i3z}}{z(z^2+1)} dz$

zodat $\int_{\Gamma} \frac{\sin(3x)}{x(x^2+1)} dx = \text{Im} \left(\int_{\Gamma} \frac{e^{i3z}}{z(z^2+1)} dz \right)$

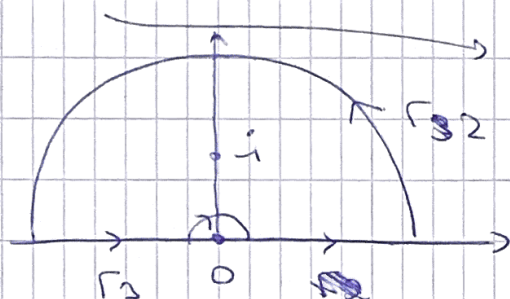
→ singuliere punten bepalen van $f(z)$:

$$z(z^2+1) = 0 \quad (\Rightarrow) \quad z=0 \vee z^2=-1$$

$$(\Rightarrow) \quad z=0 \vee z=\pm i$$

$$\text{en } z_1=0 \text{ en } z_2=i \in [\Gamma]$$

ochtens Γ :



ophefbaar bij

$$\frac{\sin(3x)}{x(x^2+1)}, \text{ we mogen}$$

dit door integreren

met mult $f(z) = \text{mult } f(z) = -1 \rightarrow$ enkelvoudige polen

→ residu's bepalen:

$$\text{res}_{z=0} f(z) = \lim_{z \rightarrow 0} \frac{e^{i3z}}{z^2+1} = \frac{1}{1} = 1$$

$$\text{res}_{z=i} f(z) = \lim_{z \rightarrow i} \frac{e^{i3z}}{(z+i)z} = \frac{e^{-3}}{2i(i)}$$

$$= \frac{e^{-3}}{-2}$$

→ via de residustelling en 25. m.m. vinden we

$$\int_{\Gamma} f(z) dz = \frac{2\pi i e^{-3}}{-2} + \pi i = \pi i (1 - e^{-3})$$

$$\text{met } \int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz$$

$$\Gamma_2: \left| \int_{\Gamma_2} \frac{e^{i3z}}{z(z^2+1)} dz \right| = \left| \int_{\Gamma_2} g(z) e^{i3z} dz \right|$$

$$\rightarrow \text{stel } \frac{1}{g(z)}$$

$$\hookrightarrow 0 \text{ als } |z| \rightarrow +\infty$$

$$= \left| \int_0^{\pi} g(Re^{i\theta}) e^{i3Re^{i\theta}} iRe^{i\theta} d\theta \right|$$

$$(0 \leq \theta \leq \pi)$$

$$\leq 2R \max |g(Re^{i\theta})| \int_0^{\pi/2} e^{-3R\sin\theta} d\theta$$

$$\leq 2R \max |g(Re^{i\theta})| \int_0^{\pi/2} e^{-\frac{2R3\theta}{\pi}} d\theta$$

$$= \left| \max |g(Re^{i\theta})| \frac{\pi}{3 \cdot (2R)} (1 - e^{-3R}) \right| \rightarrow 0$$

$$0 \leq \theta \leq \frac{\pi}{2} : \sin \theta \geq \frac{2\theta}{\pi}$$

$$\Rightarrow e^{-3R\sin\theta} \leq e^{-\frac{6R\theta}{\pi}}$$

so that

$$\int_{\Gamma_R} f(z) dz = \pi i (1 - e^{-3})$$

$$\rightarrow \int_{-\infty}^{+\infty} f(z) dz = \pi i (1 - e^{-3})$$

$$\lim_{\rightarrow} \int_{-\infty}^{+\infty} \frac{\sin(3x)}{x(x^2+1)} dx = \lim(\pi i (1 - e^{-3})) = \pi (1 - e^{-3})$$

$$\int_0^{+\infty} \frac{\ln(x)}{1+x^3} dx$$

we betrachten $\int_{\gamma} \frac{\ln(z)}{1+z^3} dz$ mit $\ln(z) = \ln|z| + i \arg(z)$

→ singuläre punten:

$$1+z^3=0 \Leftrightarrow z^3=-1$$

$$\Leftrightarrow z = e^{i\pi/3} \rightarrow \text{unkeltpool}$$

an 0 ($\ln(0)$!) → essentiiel

→ residu berechnen

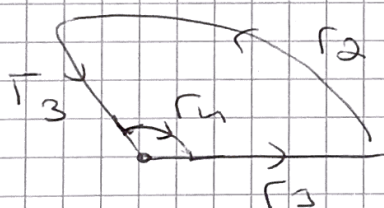
$$\text{res}_{z=e^{i\pi/3}} f(z) = \lim_{z \rightarrow e^{i\pi/3}} \left(\frac{(z-e^{i\pi/3})^2 \ln(z)}{1+z^3} \right)$$

~~$$= \lim_{z \rightarrow e^{i\pi/3}} \frac{(z-e^{i\pi/3})^2 \ln(z)}{(z-e^{i\pi/3})^2 (z^2 + e^{i\pi/3} z + e^{2i\pi/3})}$$~~

$$= \lim_{z \rightarrow e^{i\pi/3}} \frac{(z-e^{i\pi/3}) \ln(z)}{(z-e^{i\pi/3}) (z^2 + e^{i\pi/3} z + e^{2i\pi/3})}$$

$$= \lim_{z \rightarrow e^{i\pi/3}} \frac{\ln(z)}{(z^2 + e^{i\pi/3} z + e^{2i\pi/3})}$$

$$= \frac{i\pi/3}{3e^{2\pi i/3}}$$



$$\Gamma_2: \left| \int_{\gamma_2} \frac{\ln(z)}{1+z^3} dz \right| = \left| \int_0^{2\pi/3} \frac{\ln(R e^{it}) R e^{it} dt}{1+R^3 e^{3it}} \right|$$

$$\leq \int_0^{2\pi/3} \frac{|\ln(R) + it| R dt}{|1+R^3 e^{3it}|}$$

$$\leq \frac{\sqrt{\ln(R)^2 + \frac{4\pi^2}{9}}}{R^3+1} R \ell(\gamma_2) \xrightarrow{R \rightarrow +\infty} 0$$

$$\Gamma_3: \left| \int_{\gamma_3} \frac{\ln(z)}{1+z^3} dz \right| \rightarrow 0 \quad \text{analog}$$

$$\boxed{r_3}: z(t) = t e^{i \frac{2\pi}{3}} \quad t \in \mathbb{R} \setminus \{0\}$$

$$\int_{r_3} \frac{\ln(z)}{1+z^3} dz = -e^{i \frac{2\pi}{3}} \int_0^R \frac{\ln(t)}{1+t^3} dt$$

$$-e^{i \frac{2\pi}{3}} \int_0^R \frac{dt}{1+t^3} dt$$

($\ln(z)$ gerade)

Gekannt wie Residuumsatz

$$\frac{2\pi^2}{9} e^{i \frac{\pi}{3}} = (1 - e^{i \frac{2\pi}{3}}) \int_0^{+\infty} \frac{\ln(x)}{1+x^3} dx$$

$$- \frac{2\pi i}{3} e^{i \frac{2\pi}{3}} \int_0^{+\infty} \frac{dx}{1+x^3}$$

$$\Rightarrow \frac{2\pi^2}{9} = (e^{-i \frac{\pi}{3}} - e^{i \frac{\pi}{3}}) \int_0^{+\infty} \frac{\ln(x)}{1+x^3} dx$$

$$- \frac{2\pi i}{3} e^{i \frac{2\pi}{3}} \int_0^{+\infty} \frac{dx}{1+x^3} \quad | e^{i \frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$\Rightarrow \frac{-2\pi^2}{9} = \sqrt{3} i \int_0^{+\infty} \frac{\ln(x)}{1+x^3} dx$$

$$- \left(\frac{\pi}{3} i - \frac{\sqrt{3}\pi}{3} \right) \int_0^{+\infty} \frac{dx}{1+x^3}$$

$$(1): \frac{2\pi}{3} = \sqrt{3} \int_0^{+\infty} \frac{dx}{1+x^3} \Rightarrow \int_0^{+\infty} \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}}$$

$$(2) \quad 0 = \sqrt{3} \int_0^{+\infty} \frac{\ln(x)}{1+x^3} dx + \frac{\pi}{3} \int_0^{+\infty} \frac{dx}{1+x^3}$$

$$= \sqrt{3} \int_0^{+\infty} \frac{\ln(x)}{1+x^3} dx + \frac{2\pi^2}{9\sqrt{3}}$$

$$\Rightarrow \int_0^{+\infty} \frac{\ln(x)}{1+x^3} dx = \frac{-2\pi^2}{27}$$

Oefening 3.2.9.:

gegeven: $|f(z)| = |g(z)|, \forall z \in \partial B(0,1)$

$$\text{mult}_{z=a} f(z) = \text{mult}_{z=a} g(z), \forall a \in B(0,1)$$

oplossing

~~$\forall z \in \partial B(0,1)$~~

~~$\forall z \in B(0,1)$~~

~~stel~~

Beschouw $\frac{f}{g}$. Is $\frac{f}{g}$ holomorf op $\bar{B}(0,1)$?

$\frac{f}{g}$ is niet holomorf in de nulpunten van g

we willen aantonen dat in de nulpunten van g , dat $\frac{f}{g}$ een ophefbare singulariteit is

↳ De nulpunten van g zijn geïsoleerd, tenzij $g = 0$ (eenheidsfunctie)

$$\Rightarrow f = 0 \text{ op } \partial B(0,1)$$

$$\Rightarrow f = 0 \text{ op } B(0,1) \text{ (eenheidsfunctie)}$$

$$\Rightarrow f = cg$$

stel z_0 is een nulpunt van g binnen $B(0,1)$

$$\rightarrow \text{mult}_{z=z_0} \frac{f(z)}{g(z)} = \text{mult}_{z=z_0} f(z) - \text{mult}_{z=z_0} g(z) = 0$$

→ z is een ophefbaar singulier punt van $\frac{f}{g}$

stel z_0 is een nulpunt van g op $\partial B(0,1)$

$$\rightarrow \text{mult}_{z=z_0} \frac{f(z)}{g(z)} = \text{mult}_{z=z_0} f(z) - \text{mult}_{z=z_0} g(z)$$

→ z_0 is ofwel een pool van $\frac{f}{g}$ ofwel een ophefbaar singulier punt van $\frac{f}{g}$,

$$\forall B(0,1)$$

maar omdat $|f(z)| = |g(z)|$ is,

$\lim_{z \rightarrow +\infty} \frac{f(z)}{g(z)} \neq \infty$ zodat z_0 geen pool is

maar een ophefbaar singulier punt van f/g

→ f/g kan uitgereïd worden tot een holomorfe functie op $\bar{B}(0,1)$

$$\hookrightarrow \left| \frac{f}{g} \right| \leq 1 \text{ op } \partial B(0,1)$$

$$\hookrightarrow \left| f/g \right| \leq 1 \text{ op } B(0,1), \text{ analoges is } |g/g| \leq 1 \text{ op } B(0,1)$$

$$\Rightarrow \left| \frac{f}{g} \right| = 1$$

→ $\frac{f}{g}$ is holomorf en $|\frac{f}{g}|$ bereikt een maximum op $B(0,1)_{\mathbb{C}}$

zodat $\frac{f}{g} = c$ met $|c| = 1$ op $\overline{B(0,1)}$
(maximum principe)

→ $f = cg$