~~xxxxxxxxxxxx~~ $\boxed{3.6 \text{ p } 54} \to 173$

oefening 1:

a) scalairveld: $f(x,y) = y^2$

gladde kromme Γ: eenheidscirkel

- parametervoorstelling: $\vec{\varphi}(t) = (\cos t, \sin t)$ $\quad (t \in [0, 2\pi])$

$\vec{\varphi}'(t) = (-\sin t, \cos t)$

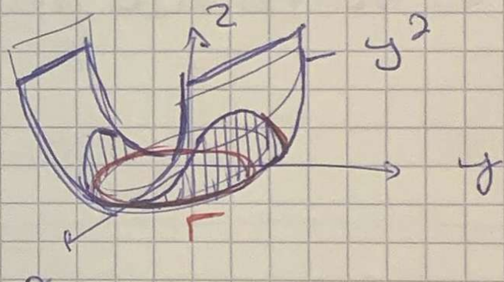
$\|\vec{\varphi}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

lijnintegraal:

$$\int_\Gamma f \, ds = \int_0^{2\pi} \sin^2 t \cdot 1 \, dt = -\int_0^{2\pi} \frac{\cos 2t - 1}{2} dt$$

$$= -\int_0^{2\pi} \frac{\cos 2t}{2} dt + \int_0^{2\pi} \frac{1}{2} dt$$

$$= \left[\frac{-\sin 2t}{4} \right]_0^{2\pi} + \left[\frac{t}{2} \right]_0^{2\pi} = \frac{2\pi}{2} = \pi \quad ✓$$



b) scalairveld: $f(x,y,z) = z$

parametervoorstelling Γ:

$\vec{\varphi} : [0, a] \to \mathbb{R}^3 : t \mapsto (t\cos t, t\sin t, t)$

$\vec{\varphi}'(t) = (\cos t - t\sin t, \sin t + t\cos t, 1)$

$$\|\vec{\varphi}'(t)\| = \sqrt{\cos^2 t + t^2 \sin^2 t - 2t\sin t\cos t + 1 + \sin^2 t + \cos^2 t\, t^2 + 2t\sin t\cos t}$$

$$= \sqrt{1 + t^2 + 1} = \sqrt{2 + t^2}$$

lijnintegraal:

$$\int_\Gamma z \, ds = \int_0^a t \sqrt{2+t^2} \, dt \quad \text{stel } 2 + t^2 = x \Rightarrow 2t\,dt = dx$$

$$= \frac{1}{2}\int_2^{2+a^2} \sqrt{x}\, dx = \frac{1}{2}\left[\frac{x^{3/2}}{3/2} \right]_{a^2}^{2+a^2} = \frac{(2+a^2)^{3/2}}{3} + \frac{a^3}{3} \quad ✓$$

e) scalairveld: $P(x,y,z) = x^2+y^2+z^2$

parametervoorstelling Γ:

$\vec{\varphi}: [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (a\cos t, a\sin t, bt)$

$\vec{\varphi}'(t) = (-a\sin t, a\cos t, b)$

$\|\vec{\varphi}'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$

$= \sqrt{a^2+b^2}$

lijnintegraal:

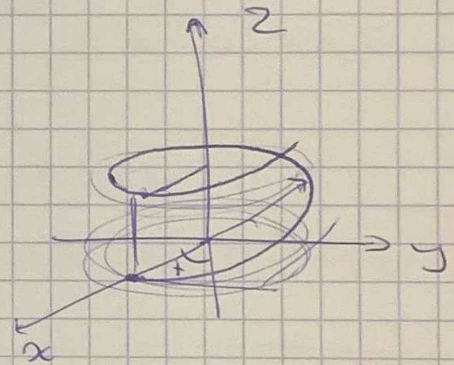
$$\int_\Gamma (x^2+y^2+z^2)\,ds = \int_0^{2\pi} (a^2\cos^2 t + a^2\sin^2 t + b^2t^2)\sqrt{a^2+b^2}\,dt$$

$$= \int_0^{2\pi} (a^2+b^2t^2)\sqrt{a^2+b^2}\,dt$$

$$= \int_0^{2\pi} a^2\sqrt{a^2+b^2}\,dt + \int_0^{2\pi} b^2\sqrt{a^2+b^2}\,t^2\,dt$$

$$= a^2\sqrt{a^2+b^2}\,2\pi + \frac{b^2\sqrt{a^2+b^2}}{3}\cdot(2\pi)^3$$

$$= \sqrt{a^2+b^2}\left(a^2 2\pi + \frac{b^2 8\pi^3}{3}\right)$$ ✓



oefening 2 a) P54 H3

vectorveld: $f(x,y,z) = (-y, x, -z/2)$

parametervoorstelling Γ:

$\vec{\varphi} : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, 2t)$

$(1,0,0) \to (1,0,4\pi)$

$$\begin{cases} \cos t = 1 \\ \sin t = 0 \\ 2t = 0 \end{cases} \Rightarrow \begin{cases} \cos t = 1 \\ \sin t = 0 \\ \boxed{t = 0} \end{cases}$$

$$\begin{cases} \cos t = 1 \\ \sin t = 0 \\ 2t = 4\pi \end{cases} \Rightarrow \begin{cases} \cos 2\pi = 1 \\ \sin 2\pi = 0 \\ t = 2\pi \end{cases}$$

$\varphi'(t) = (-\sin t, \cos t, 2)$

lijnintegraal:

$$\int_\Gamma f(x,y,z) \cdot d\vec{r} = \int_0^{2\pi} (-\sin t, \cos t, -t)(-\sin t, \cos t, 2)\, dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t - 2t \, dt = \int_0^{2\pi} 1 - 2t \, dt$$

$$= \left[t - t^2 \right]_0^{2\pi} = 2\pi - 4\pi^2$$

---

b) parametervoorstelling:

$\vec{\varphi} : [0, 1] \to \mathbb{R}^2 : t \mapsto (t, 3t^2)$

$\vec{\varphi}'(t) = (1, 6t)$

lijnintegraal

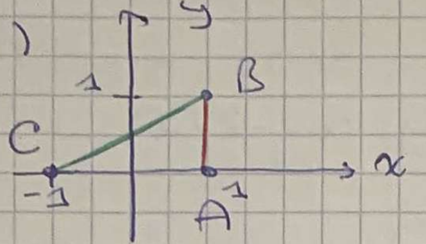
$$\int_\Gamma (x^2 y, x^2 - y^2)\, d\vec{r} = \int_0^1 (t^2 \cdot 3t^2) \cdot 1 + (t^2 - 9t^4) 6t \, dt$$

$$= \int_0^1 3t^4 + 6t^3 - 54t^5 \, dt$$

$$= \left[\frac{3t^5}{5} + \frac{3}{2}t^4 - 9t^6 \right]_0^1 = \frac{3}{5} + \frac{3}{2} - 9$$

$$= -\frac{69}{10}$$

d)



van A → B parametervoorstelling:

$$\vec{\varphi}_1 : [0,1] \to \mathbb{R}^2 : t \mapsto ((1-t)1 + t\cdot 1,\ (1-t)0 + t\cdot 1)$$
$$= (1-t+t,\ t) = (1,t)$$

van B → C parametervoorstelling:

$$\vec{\varphi}_2 : [0,1] \to \mathbb{R}^2 : t \mapsto ((1-t)1 - t,\ (1-t)1 - 0)$$
$$= (1-2t,\ 1-t)$$

$$\vec{\varphi}_1'(t) = (0,1)$$

$$\vec{\varphi}_2'(t) = (-2,-1)$$

lijnintegraal

$$\int_\Gamma \vec{f}\cdot d\vec{r} = \int_0^1 (1-t^3)0 + 1\cdot 1\, dt$$
$$+ \int_0^1 (1-2t-(1-t)^3)(-2) - 1(1-2t)^3\, dt$$

$$= 1 + \int_0^1 \cancel{1}-2+4t+2(t+1)^3 - (1-2t)^3\, dt$$

$$= 1 + \left[-2t+2t^2 - \frac{2}{4}(t+1)^4 + \frac{2(1-2t)^4}{4}\right]_0^1$$

$$= 1 + \left[-2t+2t^2 - \frac{1}{2}(1-t)^4 + \frac{1}{2}(1-2t)^4\right]_0^1$$

$$= 1 + \left(-2+2+\frac{1}{2}\right) - \left(-\frac{1}{2}+\frac{1}{2}\right)$$

$$= 1 + \frac{1}{2} = \frac{3}{2}$$

oefening 1: c) H5: oppervlakte integralen

- parametervoorstelling $x^2+y^2+z^2=R^2$: Σ

$$\varphi: K \to \mathbb{R}^3 : (\theta,\psi) \mapsto (R\cos\theta\sin\psi,\ R\sin\theta\sin\psi,\ R\cos\psi)$$

$$K = \{(\theta,\psi)\in\mathbb{R}^2 \mid 0\le\theta\le 2\pi,\ 0\le\psi\le \pi/2\}$$

$$\frac{\partial\varphi}{\partial\theta}\times\frac{\partial\varphi}{\partial\psi} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -R\sin\theta\sin\psi & R\cos\theta\sin\psi & 0 \\ R\cos\theta\cos\psi & R\sin\theta\cos\psi & -R\sin\psi \end{vmatrix}$$

$$= (-R^2\cos\theta\sin^2\psi,\ -R^2\sin\theta\sin^2\psi,\ -R^2\sin\psi\cos\psi)$$

$$\left\|\frac{\partial\varphi}{\partial\theta}\times\frac{\partial\varphi}{\partial\psi}\right\| = \sqrt{R^4\sin^4\psi + R^4\sin^2\psi\cos^2\psi}$$

$$= \sqrt{R^4\sin^2\psi} = R^2\sin\psi$$

!!

- oppervlakte-integraal:

$$\iint_\Sigma \frac{d\sigma}{\sqrt{x^2+y^2+(z+R)^2}} = \iint_K \frac{d\theta\,d\psi\cdot R^2\sin\psi}{\sqrt{R^2\cos^2\theta\sin^2\psi + R^2\sin^2\theta\sin^2\psi + (R\cos\psi+R)^2}}$$

$$= \iint_K \frac{d\theta\,d\psi\ R^2\sin\psi}{\sqrt{R^2\sin^2\psi + R^2\cos^2\psi + 2R^2\cos\psi + R^2}}$$

$$= \iint_K \frac{d\theta\,d\psi}{\sqrt{R^2 + 2R^2\cos\psi + R^2}} = \iint_K \frac{d\theta\,d\psi}{\sqrt{2R^2(1+\cos\psi)}}$$

$$= \iint_K \frac{d\theta\,d\psi}{\sqrt{2}R\cdot\sqrt{1+\cos\psi}} \qquad \cos\psi = 2\cos^2\frac{\psi}{2} - 1$$

$$= \iint_K \frac{d\theta\,d\psi}{\sqrt{2}R\sqrt{2\cdot\cos^2\left(\frac{\psi}{2}\right)}} = \iint_K \frac{d\theta\,d\psi}{2R\cos\left(\frac{\psi}{2}\right)}$$

$$= \iint_K \frac{\sec\left(\frac{\psi}{2}\right)\left(\tan\left(\frac{\psi}{2}\right)+\sec\left(\frac{\psi}{2}\right)\right)d\theta\,d\psi}{2R\left(\tan\left(\frac{\psi}{2}\right)+\sec\left(\frac{\psi}{2}\right)\right)}$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \frac{\sec\left(\frac{\psi}{2}\right)\left(\tan\left(\frac{\psi}{2}\right)+\sec\left(\frac{\psi}{2}\right)\right)}{2R\left(\tan\left(\frac{\psi}{2}\right)+\sec\left(\frac{\psi}{2}\right)\right)}\, d\psi$$

stel $\tan\left(\frac{\psi}{2}\right)+\sec\left(\frac{\psi}{2}\right) = u$

$$\Rightarrow \left(\sec^2\left(\frac{\psi}{2}\right)\cdot\tfrac{1}{2} + (+1)\sec^2\left(\frac{\psi}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right)\cdot\frac{1}{2}\right)d\psi = du$$

$$\Rightarrow \frac{1}{2}\left(\sec^2\frac{\psi}{2} + \sec\frac{\psi}{2}\cdot\tan\frac{\psi}{2}\right)d\psi = du$$

als $\psi\to\frac{\pi}{2}$, dan $u\to 1+\sqrt{2}$

als $\psi\to 0$, dan $u\to 1$

$$= \int_0^{2\pi} d\theta \int_1^{1+\sqrt{2}} \frac{1}{2R}\frac{1}{u}\,du = \int_0^{2\pi} \frac{\ln|1+\sqrt{2}|}{2R}\,d\theta = \frac{\ln|1+\sqrt{2}|\cdot 2\pi}{2R}$$

$$= \iint_K \frac{R \sin \psi}{2R \cos \frac{\psi}{2}} \, d\theta \, d\psi = \iint_K \frac{R \, 2 \sin \frac{\psi}{2} \cos \frac{\psi}{2}}{2 \cos \frac{\psi}{2}} \, d\theta \, d\psi$$

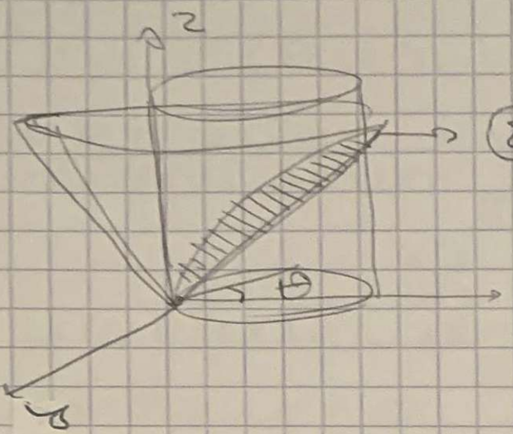
$$= R \iint_K \cancel{2} \sin \frac{\psi}{2} \, d\theta \, d\psi = R \int_0^{2\pi} d\theta \int_0^{\pi/2} \cancel{2} \sin \frac{\psi}{2} \, d\psi$$

$$= R\cancel{2} \int_0^{2\pi} d\theta \left[-\cancel{2}\cos \frac{\psi}{2} \right]_0^{\pi/2} = \cancel{2}R \int_0^{2\pi} -\frac{\sqrt{2}}{2} + 1 \, d\theta$$

$$= \cancel{2}R \left[-\frac{\sqrt{2}}{2} \theta + \theta \right]_0^{2\pi} = \cancel{2}R \left(-\frac{2\pi \sqrt{2}}{2} + 2\pi \right)$$

$$= \cancel{2}R \left(-\sqrt{2}\pi + 2\pi \right) = \cancel{2}\pi R \left(-\sqrt{2} + 2 \right)$$

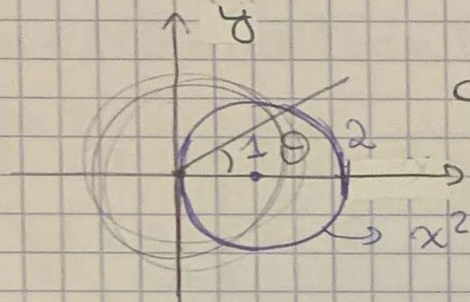
oefening 1. d) H5



- parametervoorstelling Σ:
$\varphi: K \to \mathbb{R}^3: (\theta, r) \mapsto (r\cos\theta, r\sin\theta, r)$

Σ ↳ K bepalen: $x^2+y^2 = 2x$
$\to r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$
$\Rightarrow r^2 = 2r\cos\theta$
$\Rightarrow r = 2\cos\theta$

bovenaanzicht



$\to x^2+y^2 = 2x$

dus $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$$K = \{(\theta, r) \in \mathbb{R}^2 \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\cos\theta\}$$

$$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = (r\cos\theta, r\sin\theta, -r)$$

$$\left\| \frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} \right\| = \sqrt{r^2+r^2} = \sqrt{2}r$$

- oppervlakte-integraal

$$\iint_\Sigma (x^4 - y^4 + y^2z^2 - z^2x^2 + 1)\,d\sigma$$

$$= \iint_K (r^4\cos^4\theta - r^4\sin^4\theta + r^4\sin^2\theta - r^4\cos^2\theta + 1)\underbrace{\ }_{\sqrt{2}r}\,d\theta\,dr$$

$$= \iint_K r^4\cos^2\theta(\cos^2\theta - 1) + r^4\sin^2\theta(1-\sin^2\theta) + 1)\sqrt{2}r\,d\theta\,dr$$

$$= \iint_K (r^4\cos^2\theta(-\sin^2\theta) + r^4\sin^2\theta\cos^2\theta + 1)\sqrt{2}r\,d\theta\,dr$$

$$= \iint_K \sqrt{2}r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{\sqrt{2}}{2}[r^2]_0^{2\cos\theta}\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\sqrt{2}}{2} 4\cos^2\theta\,d\theta = \int_{-\pi/2}^{\pi/2} 2\sqrt{2}\cos^2\theta\,d\theta$$

$2\cos^2\theta - 1 = \cos 2\theta$
$\Leftrightarrow 2\cos^2\theta = \cos 2\theta + 1$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{2}(\cos 2\theta + 1)\,d\theta = \left[\frac{\sqrt{2}}{2}\sin 2\theta + \sqrt{2}\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2}$$

$$= \frac{2\sqrt{2}\pi}{2} = \sqrt{2}\pi.$$

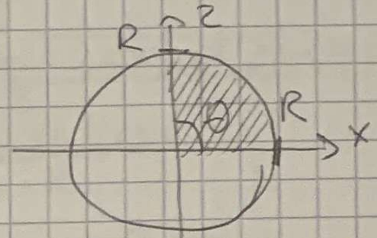
oefening 2: H5

a) parametervoorstelling van cilinder $x^2+y^2=R^2$:

$\varphi: K \to \mathbb{R}^3 : (\theta, t) \mapsto (R\cos\theta, R\sin\theta, t)$

↳ K bepalen: in $x^2+z^2=R^2$ onze parameters substitueren:

$$R^2\cos^2\theta + t^2 = R^2$$
$$\Rightarrow t = \pm R\sqrt{1-\cos^2\theta}$$
$$\Rightarrow t = \pm R|\sin\theta|$$



we kiezen voor $t>0$, dus:

$R|\sin\theta| > t > 0$

nu we willen $\sin\theta > 0 \Rightarrow \theta \in]0, \frac{\pi}{2}[$?

dus $K = \{(\theta, t) \in \mathbb{R}^2 \mid 0<\theta<\frac{\pi}{2}, 0<t<R\sin\theta\}$ om herhaling tegen te gaan

↳ $\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial t} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (R\cos\theta, R\sin\theta, 0)$

$\left\| \frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial t} \right\| = R$

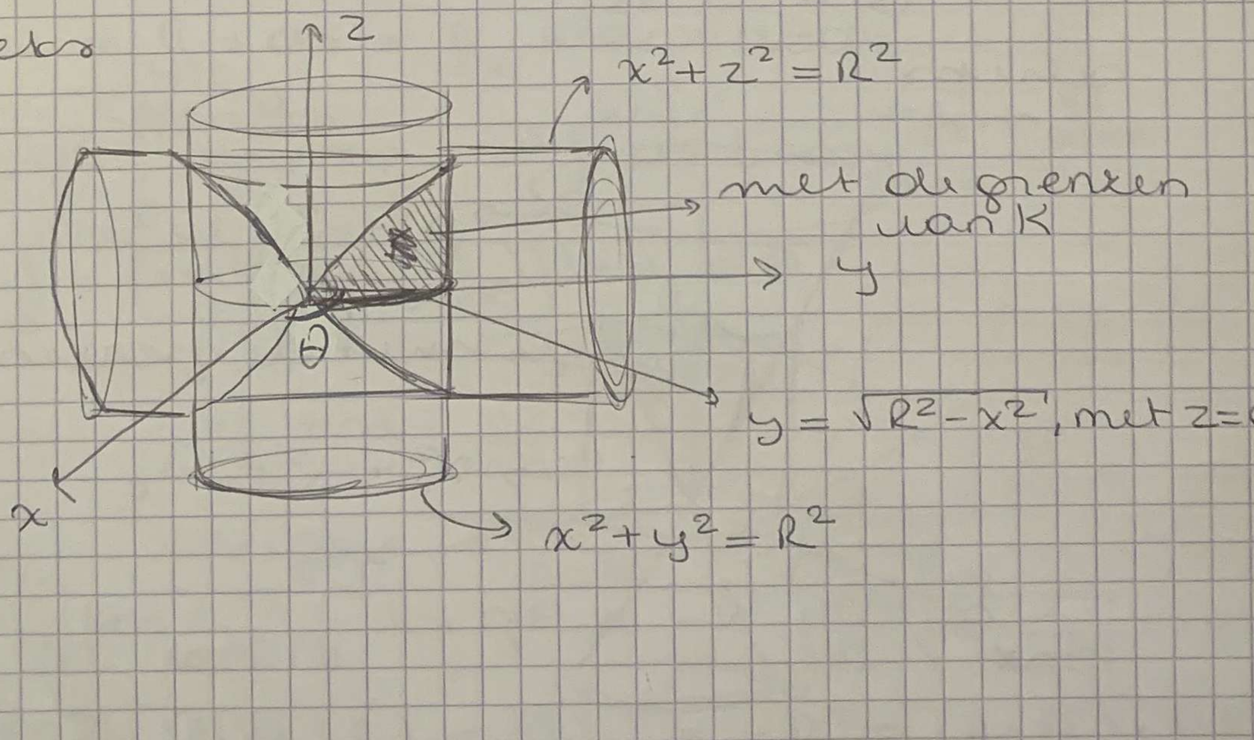
- oppervlakteintegraal:

$$\iint_\Sigma 1\, ds = 16\iint_K R\, dt\, d\theta = 16\int_0^{\pi/2} d\theta \int_0^{R\sin\theta} R\, dt$$

$$= 16\int_0^{\pi/2} [Rt]_{t=0}^{t=R\sin\theta}\, d\theta = 16\int_0^{\pi/2} R^2\sin\theta\, d\theta$$

$$= 16R^2[-\cos\theta]_0^{\pi/2} = 16R^2(0-(-1)) = 16R^2 \checkmark$$

schets



oefening 2: b)

- parametervoorstelling kegelmantel $x^2+y^2=3z^2$

$\varphi : K \to \mathbb{R}^3 : (\theta, r) \mapsto \left(r\cos\theta,\ r\sin\theta,\ \frac{r}{\sqrt{3}}\right)$

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2 = 3z^2 \Rightarrow z = r/\sqrt{3}$$

↳ K bepalen : $x^2+y^2 = 4y$

substitutie: $r^2\cos^2\theta + r^2\sin^2\theta = 4\sin\theta \cdot r$

$\Rightarrow r^2 = 4r\sin\theta$

$\Rightarrow r = 4\sin\theta$ $\quad (r>0!)$

$\sin\theta \geq 0 \Rightarrow \theta \in [0, \pi/2]$

$K = \{(\theta, r) \in \mathbb{R}^2 \mid 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\sin\theta\}$

↳ $\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1/\sqrt{3} \end{vmatrix} = \left(\frac{r}{\sqrt{3}}\cos\theta, \frac{r}{\sqrt{3}}\sin\theta, -r\right)$

$\left\| \frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} \right\| = \sqrt{\frac{r^2}{3} + r^2} = \sqrt{\frac{r^2+3r^2}{3}} = \frac{2r}{\sqrt{3}}$

- oppervlakte-integraal:

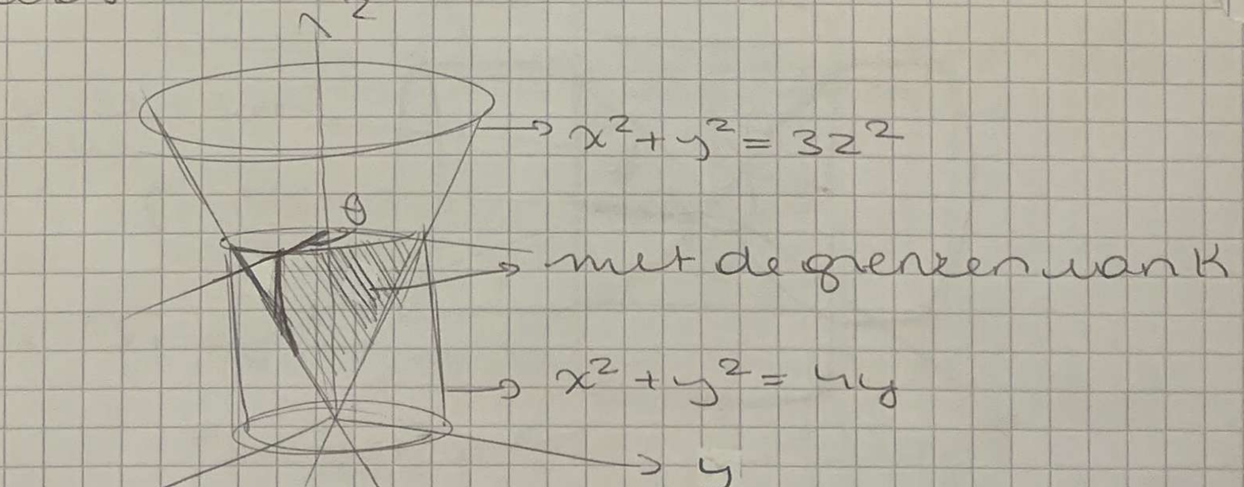
$\text{opp}(\Sigma) = \iint_\Sigma 1\, d\sigma = 4\iint_K \frac{2r}{\sqrt{3}}\, d\theta\, dr$

$= 4\int_0^{\pi/2} d\theta \int_0^{2\sin\theta} \frac{2r}{\sqrt{3}}\, dr = \frac{4}{\sqrt{3}} \int_0^{\pi/2} [r^2]_{r=0}^{r=2\sin\theta}\, dr$

$= \frac{4}{\sqrt{3}} \int_0^{\pi/2} 4\sin^2\theta\, d\theta = -\frac{16}{\sqrt{3}} [\sin(2\theta)]_0^{\frac{\pi}{2}} + \frac{8}{\sqrt{3}} [4\theta]_0^{\pi/2}$

$= -0 + \frac{8}{\sqrt{3}} \frac{4}{2}\pi = \frac{16\pi}{\sqrt{3}}$ ✓

schets

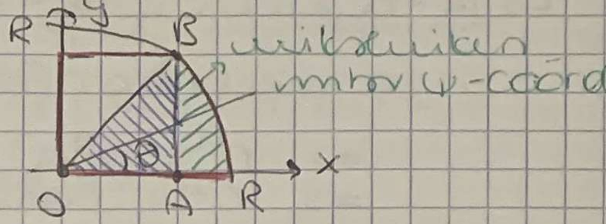


oefening 2 (c) H5

- parametervoorstelling bolopppervlak $x^2+y^2+z^2=R^2$ $(z \geq 0)$

$\varphi: K \to \mathbb{R}^3 : (\theta, \psi) \mapsto (R\cos\theta\sin\psi, R\sin\theta\sin\psi, R\cos\psi)$

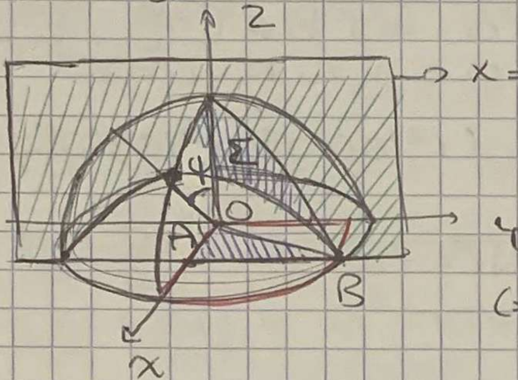
↳ K bepalen



projectie op xy-vlak:

$\frac{R}{\sqrt{2}} = R\cos\theta$

$\iff \frac{\sqrt{2}}{2} = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$ dus $\theta \in [0, \frac{\pi}{4}]$



→ $x = \frac{R}{\sqrt{2}}$ → we bepalen ψ zodat het x-coördinaat van ψ $R/\sqrt{2}$ is:

$R\cos\theta\sin\psi = \frac{R}{\sqrt{2}}$

$\iff \cos\theta\sin\psi = \frac{\sqrt{2}}{2}$

$\Rightarrow \sin\psi = \frac{\sqrt{2}}{2\cos\theta}$

$\Rightarrow \psi = \operatorname{Bgsin}\frac{\sqrt{2}}{2\cos\theta}$

dus $K = \{(\theta,\psi) \in \mathbb{R}^2 \mid 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq \psi \leq \operatorname{Bgsin}\frac{\sqrt{2}}{2\cos\theta}\}$

$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial\psi} = (-R^2\cos\theta\sin^2\psi, -R^2\sin\theta\sin^2\psi, -R^2\cos\psi\sin\psi)$

$\left\| \frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial\psi} \right\| = R^2\sin\psi$

- oppervlakte-integraal:

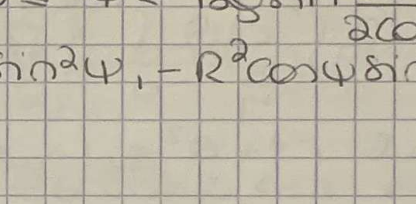
$\operatorname{opp}(\Sigma) = \iint_\Sigma 1 \, d\sigma = \iint_K R^2\sin\psi \, d\theta \, d\psi$

$= R^2 \int_0^{\pi/4} d\theta \int_0^{\operatorname{Bgsin}\frac{\sqrt{2}}{2\cos\theta}} \sin\psi \, d\psi = -R^2 \int_0^{\pi/4} d\theta \, [\cos\psi]_{\psi=0}^{\psi=\operatorname{Bgsin}\frac{\sqrt{2}}{2\cos\theta}}$

$= -R^2 \int_0^{\pi/4} \boxed{\cos\left(\operatorname{Bgsin}\left(\frac{\sqrt{2}}{2\cos\theta}\right)\right)} - 1 \, d\theta$

↳ stel $\operatorname{Bgsin}\frac{\sqrt{2}}{2\cos\theta} = \alpha$

$\Rightarrow \sin\alpha = \frac{\sqrt{2}}{2\cos\theta}$



$\Rightarrow \cos\alpha = \sqrt{1 - \frac{1}{2\cos^2\theta}}$

$= -R^2 \int_0^{\pi/4} \sqrt{\left(2 - \frac{1}{\cos^2\theta}\right)\frac{1}{2}} - 1 \, d\theta = -R^2 \int_0^{\pi/4} \frac{1}{\sqrt{2}}\sqrt{2 - \frac{1}{\cos^2\theta}} - 1 \, d\theta$

$= -R^2\left(\frac{\pi}{2} \cdot \frac{\sqrt{2}}{2}(\sqrt{2}-1) - \frac{\pi}{4}\right) = -\frac{R^2\pi}{4}(2-\sqrt{2}) + \frac{\pi}{4}R^2$

$= \frac{\pi}{4}(+R^2(-2+\sqrt{2}+1)) = \frac{\pi}{4}R^2(\sqrt{2}-1)$ ✓

d) - parametervoorstelling $x^2+y^2 = 2az$ $(a>0)$

$\varphi : K \to \mathbb{R}^3 : (\theta, r) \mapsto (r\cos\theta, r\sin\theta, \frac{r^2}{2a})$

$r^2 = 2az \Leftrightarrow z = \frac{r^2}{2a}$

↳ K - bepalen : $(x^2+y^2)^2 = a^2(x^2-y^2)$

$\to (r^2\cos^2\theta + r^2\sin^2\theta)^2 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$

$\Rightarrow r^4 = r^2a^2\cos 2\theta$

$\Rightarrow r^2 = ra^2\cos 2\theta \Rightarrow r = a\sqrt{\cos 2\theta}$

$\to 0 \le r \le a\sqrt{\cos 2\theta}$

↳ $\cos 2\theta \ge 0 \Rightarrow \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$

$K = \{(\theta, r) \in \mathbb{R}^2 \mid -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, 0 \le r \le a\sqrt{\cos 2\theta}\}$

$$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & r/a \end{vmatrix} = (r^2/a\cos\theta, r^2/a\sin\theta, -r)$$

$$\left\| \frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} \right\| = \sqrt{\frac{r^4}{a^2} + r^2} = \frac{r}{a}\sqrt{r^2+a^2}$$

- oppervlakte - integraal

$$\text{opp}(\Sigma) = 2\iint_\Sigma 1 \, d\sigma = 2\iint_K \frac{r}{a}\sqrt{r^2+a^2}\, d\theta\, dr$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^{a\sqrt{\cos 2\theta}} \frac{r}{a}\sqrt{r^2+a^2}\, dr$$

stel $r^2 + a^2 = u \Rightarrow 2r\,dr = du$

als $r \to a\sqrt{\cos 2\theta}$ dan $r^2 + a^2 \to a^2\cos 2\theta + a^2$

als $r \to 0$ dan $r^2 + a^2 \to a^2$

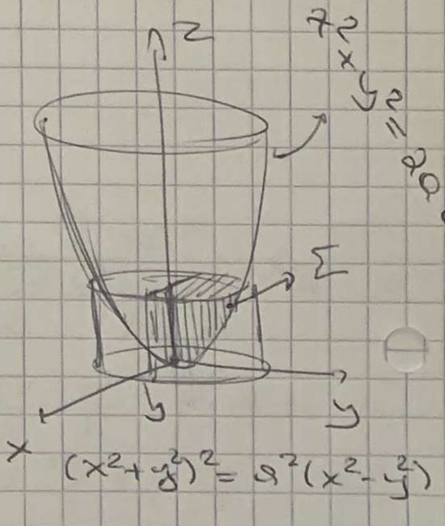
$$= 2\int_{-\pi/4}^{\pi/4} d\theta \int_{a^2}^{a^2\cos 2\theta + a^2} \frac{1}{2a}\sqrt{u}\, du = \frac{1}{a}\int_{-\pi/4}^{\pi/4} d\theta \left[\frac{u^{3/2}}{3/2}\right]_{u=a^2}^{u=a^2\cos 2\theta + a^2}$$

$$= \frac{1}{a}\int_{-\pi/4}^{\pi/4} \frac{(a^2 + a^2\cos 2\theta)^{3/2}}{3/2} - \frac{(a^2)^{3/2}}{3/2}\, d\theta \quad \Big) \; \cos 2\theta = 2\cos^2\theta - 1$$

$$= \frac{2}{3a}\int_{-\pi/4}^{\pi/4} a^3(\cos^3\theta)2^{3/2} - a^3\, d\theta$$

$$= \frac{2a^2}{3}\int_{-\pi/4}^{\pi/4} 2\sqrt{2}\cos^3\theta - 1\, d\theta$$

$$= \ldots = \frac{a^2}{9}(20 - 3\pi)\, d\theta$$



oefening 2: e) H5

- parametervoorstelling $x^2+y^2+z^2 = a^2 \ (a>0)$

$\varphi: K \to \mathbb{R}^3 : (\theta, \psi) \mapsto (a\cos\theta\sin\psi, a\sin\theta\sin\psi, a\cos\psi)$

↳ K bepalen: $(x^2+y^2)^2 = a^2(x^2-y^2)$

$$(a^2\sin^2\psi)^2 = a^2(a^2\cos^2\theta\sin^2\psi - a^2\sin^2\theta\sin^2\psi)$$

$$\Rightarrow a^4\sin^4\psi = a^2(a^2\sin^2\psi\cos 2\theta)$$

$$\Rightarrow \sin^2\psi = \cos 2\theta \Rightarrow \sin\psi = \pm\sqrt{\cos 2\theta}$$

↳ want $\psi \in [0, \frac{\pi}{2}]$

$$\Rightarrow \psi = \operatorname{Bgsin}(\sqrt{\cos 2\theta})$$

$$0 \leq \psi \leq \operatorname{Bgsin}(\sqrt{\cos 2\theta})$$

$$1 \geq \sqrt{\cos 2\theta} \geq 0 \Rightarrow 2\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[\Rightarrow \theta \in]-\frac{\pi}{4}, \frac{\pi}{4}[$$

$$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial\psi} = (-a\cos\theta\sin^2\psi, -a\sin\theta\sin^2\psi, a^2\cos\psi\sin\psi)$$

$$\left\|\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial\psi}\right\| = a^2\sin\psi$$

- oppervlakte-integraal:

$$\text{opp}(\Sigma) = \iint_\Sigma 1 \, d\sigma = 2\iint_K a^2\sin\psi \, d\psi \, d\theta$$

$$= 2\int_{-\pi/4}^{\pi/4} d\theta \int_0^{\operatorname{Bgsin}\sqrt{\cos 2\theta}} a^2\sin\psi \, d\psi = -2a^2\int_{-\pi/4}^{\pi/4} [\cos\psi]_{\psi=0}^{\psi=\operatorname{Bgsin}\sqrt{\cos 2\theta}} \, d\theta$$

$$= -2a^2\int_{-\pi/4}^{\pi/4} \cos(\operatorname{Bgsin}\sqrt{\cos 2\theta}) - 1 \, d\theta$$

$$= -2a^2\int_{-\pi/4}^{\pi/4} \sqrt{1-\cos(2\theta)} - 1 \, d\theta$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$$\cancel{= -2a^2\int_{-\pi/4}^{\pi/4} |\sin(2\theta)| - 1 \, d\theta}$$

$$\cancel{= -4a^2\int_0^{\pi/4} \sin(2\theta) - 1 \, d\theta}$$

$$\cancel{= -4a^2\left[-\frac{\cos(2\theta)}{2} - \theta\right]_0^{\pi/4}}$$

$$\cancel{= -4(a)^2\left(-\frac{0}{2} - \frac{\pi}{4} + \frac{1}{2}\right)} = 4a^2\left(-\frac{1}{2} + \frac{\pi}{4}\right)$$

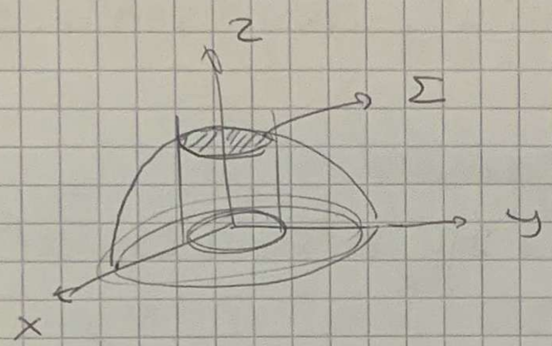
$$= -2a^2 \int_{-\pi/4}^{\pi/4} \sqrt{2}\,\sin^2\theta - 1 \, d\theta$$

$$= -4a^2 \int_{0}^{\pi/4} \sqrt{2}\sin\theta - 1 \, d\theta$$

$$= +4a^2 \left[\sqrt{2}\cos\theta + 1 \right]_0^{\pi/4}$$

$$= +4a^2 \left(\sqrt{2} \cdot \frac{\sqrt{2}}{2} + \frac{\pi}{4} - \sqrt{2} - 0 \right)$$

$$= 4a^2(1-\sqrt{2}) + a^2\pi \quad 1.2\,?$$



oefening 2 : f) +15

parametervoorstelling kromme Γ:

$\varphi: [0, \pi] \to \mathbb{R}^2 : t \mapsto (a\cos^3 t, a\sin^3 t)$

we wentelen Γ volledig om de x-as:

$opp(\Sigma) = 2\pi \int_\Gamma y \, ds$

$= 2\pi \int_0^\pi a\sin^3 t \cdot \| (-3a\cos^2 t \sin t, 3a\sin^2 t \cos t) \| \, dt$

$= 2\pi \int_0^\pi a\sin^3 t \sqrt{9a^2(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t)} \, dt$

$= 2\pi \int_0^\pi a\sin^3 t \, 3a|\cos t| \cdot \sin t \, dt$

$= 4\pi \int_0^{\pi/2} 3a^2 \sin^4 t \cos t \, dt$

stel $\sin t = u$

$\Rightarrow \cos t \, dt = du$

als $t \to \pi/2$ dan $\sin t \to 1$

als $t \to 0$ dan $\sin t \to 0$

$= 4\pi \int_0^1 3a^2 u^4 \, du = \frac{12a^2}{5} [u^5]_0^1 \pi$

$= \frac{12a^2}{5}\pi$ ✓

---

g) parametervoorstelling $x^2 + y^2 = z^2$

$\varphi: R \to \mathbb{R}^3 : (\theta, r) \mapsto (r\cos\theta, r\sin\theta, r)$ $x \geq 0$

$r^2 = z^2 \Rightarrow z = r$

↳ R bepalen: $(x^2+y^2)^2 = y^2 - x^2$

$\Rightarrow (r^2)^2 = r^2\sin^2\theta - r^2\cos^2\theta$

$\Rightarrow r^2 = \sin^2\theta - \cos^2\theta = -\cos 2\theta$

$\Rightarrow r = \sqrt{-\cos 2\theta}$

↳ $0 \leq r \leq \sqrt{-\cos 2\theta}$

$0 \geq \cos 2\theta \geq -1 \Rightarrow 2\theta \in \left] \frac{\pi}{2}, \frac{3\pi}{2} \right[\Rightarrow \theta \in \left] \frac{\pi}{4}, \frac{3\pi}{4} \right[$

$R = \{ (\theta, r) \in \mathbb{R}^2 \mid \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, 0 \leq r \leq \sqrt{-\cos 2\theta} \}$

$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = (r\cos\theta, r\sin\theta, -r)$

$\left\| \frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} \right\| = \sqrt{r^2 + r^2} = \sqrt{2r^2}$

↪

- oppervlakte integraal:

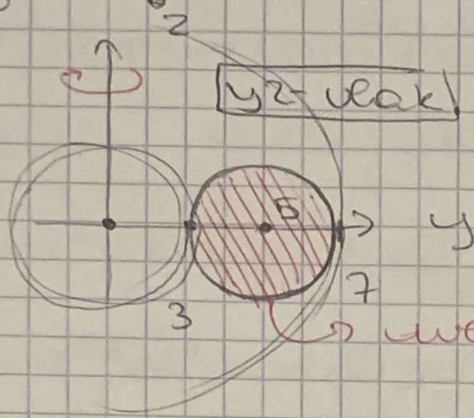
$$opp(\Sigma) = \iint_\Sigma 1 \cdot d\sigma = \cancel{2} \iint_{K \hookrightarrow x \geq 0} \sqrt{2}\, r \, dr d\theta$$

$$= \cancel{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\sqrt{-\cos 2\theta}} \sqrt{2} r \, dr = \frac{\sqrt{2}}{2} \cancel{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \left[r^2 \right]_{r=0}^{r=\sqrt{-\cos 2\theta}}$$

$$= \frac{\sqrt{2}}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} -\cos 2\theta \, d\theta = -\frac{\sqrt{2}}{\cancel{2}4} \left[\sin 2\theta \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

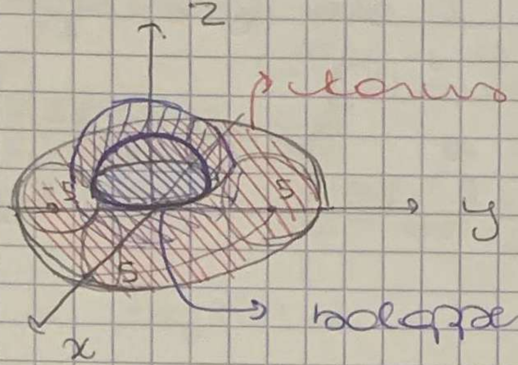
$$= -\frac{\sqrt{2}}{4} \left(\sin \frac{3\pi}{2} - \sin \frac{\pi}{2} \right) = +\frac{\sqrt{2}}{2}$$

oefening 2. h) H5



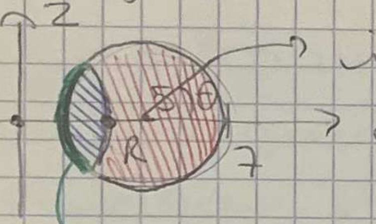
$3 < R < 7$

wentelen dit om de z-as



bol oppervlak $x^2+y^2+z^2=R^2$
$(3<R<7)$

gevraagd: oppervlak van de torus dat binnen de bol ligt.



in figuur is dit hoe de doorsnede eruit ziet

dit is dan de gevraagde kromme die we wentelen en dan de oppervlak van bepalen

parametervoorstelling cirkel: $(y-5)^2+z^2=4$

$\varphi: [0,2\pi] \to \mathbb{R}^3: \theta \mapsto (0, 2\cos\theta+5, 2\sin\theta)$

we willen deze parametrisatie in in $x^2+y^2+z^2=R^2$ zodat we weten tot wat we θ moeten beperken om het groene gedeelte te verkrijgen:

$x^2+y^2+z^2=R^2 \to 0+4\cos^2\theta+20\cos\theta+25+4\sin^2\theta=R^2$

$\Leftrightarrow 4+20\cos\theta+25=R^2 \Leftrightarrow 29+20\cos\theta=R^2$

$\Leftrightarrow 20\cos\theta = R^2-29 \Leftrightarrow \cos\theta = \frac{R^2}{20}-\frac{29}{20}$

$\Rightarrow \theta = \text{Bgcos}\left(\frac{R^2}{20}-\frac{29}{20}\right) \vee \theta = 2\pi - \text{Bgcos}\left(\frac{R^2-29}{20}\right)$

dus voor het groene krijgen we de parametervoorstelling

$\psi: \left[\text{Bgcos}\left(\frac{R^2-29}{20}\right), 2\pi-\text{Bgcos}\left(\frac{R^2-29}{20}\right)\right] \to \mathbb{R}^3: \theta \mapsto (0, 2\cos\theta+5, 2\sin\theta)$

$$\mathrm{aap}(\Sigma) = 2\pi \int_{\arccos\left(\frac{R^2-29}{20}\right)}^{2\pi-\arccos\left(\frac{R^2-29}{20}\right)} (2\cos\theta + 5)\, 2\, d\theta$$

$$\psi'(\theta) = (0, -2\sin\theta, 2\cos\theta)$$
$$\|\psi'(\theta)\| = 2$$

$$= 2\pi \int 4\cos\theta + 10\, d\theta$$

$$= 2\pi \left[4\sin\theta + 10\theta \right]_{\arccos\left(\frac{R^2-29}{20}\right)}^{2\pi-\arccos\left(\frac{R^2-29}{20}\right)}$$

$$= 4\pi \left[4\sin\theta + 10\theta \right]_{\arccos\left(\frac{R^2-29}{20}\right)}^{0}$$

$$= 16\pi \sin\left(\arccos\left(\frac{R^2-29}{20}\right)\right) + 40\pi \arccos\left(\frac{R^2-29}{20}\right)$$

oefening 2 ii) H5

$\varphi: K \to \mathbb{R}^3 : (\theta,\psi) \mapsto (a\cos\theta\sin\psi,\ a\sin\theta\sin\psi,\ a\cos\psi)$

K bepalen: $\dfrac{a^2\cos^2\theta\sin^2\psi}{a^2} + \dfrac{a^2\sin^2\theta\sin^2\psi}{b^2} = 1$

$\Rightarrow \cos^2\theta\sin^2\psi + \dfrac{a^2}{b^2}\sin^2\theta\sin^2\psi = 1$

$\Rightarrow \sin^2\psi\left(\cos^2\theta + \dfrac{a^2}{b^2}\sin^2\theta\right) = 1$

$\Rightarrow \sin^2\psi = \dfrac{1}{\cos^2\theta + \frac{a^2}{b^2}\sin^2\theta}$

$\Rightarrow \sin\psi = \dfrac{1}{\sqrt{\cos^2\theta + \frac{a^2}{b^2}\sin^2\theta}} = \alpha$

$\Rightarrow \psi = \text{Bgsin}\,\alpha$

$K = \{(\theta,\psi) \in \mathbb{R}^2 \mid 0 < \psi < \text{Bgsin}\,\alpha,\ 0 \le \theta \le \frac{\pi}{2}\}$

$\dfrac{\partial\varphi}{\partial\theta} \times \dfrac{\partial\varphi}{\partial\psi} = (-a\cos\theta\sin^2\psi,\ -a\sin\theta\sin^2\psi,\ -a^2\cos\psi\sin\psi)$

$\left\|\dfrac{\partial\varphi}{\partial\theta} \times \dfrac{\partial\varphi}{\partial\psi}\right\| = a^2\sin\psi$

oppervlakte-integraal

$4\iint_K a^2\sin\psi\, d\theta\, d\psi = 4\int_0^{\pi/2} d\theta\, \left[a^2(-1)\cos\psi\right]_{\psi=0}^{\psi=\text{Bgsin}\,\alpha}$

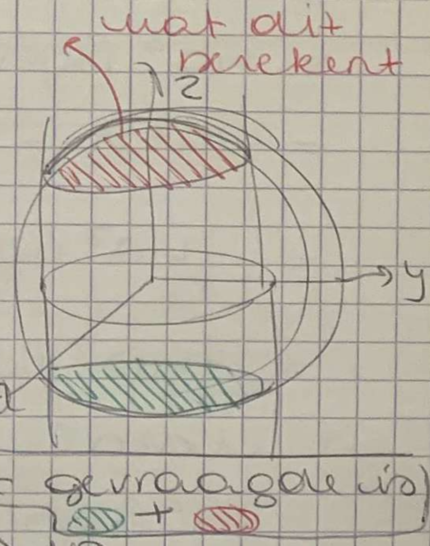
$= 4\int_0^{\pi/2} -a^2\sqrt{1-\alpha^2} + a^2\, d\theta$

$= 4a^2\int_0^{\pi/2} -\sqrt{1 - \dfrac{1}{\cos^2\theta + \frac{a^2}{b^2}\sin^2\theta}} + 1\, d\theta$

$= 4a^2\int_0^{\pi/2} -\sqrt{\dfrac{\cos^2\theta - 1 + \frac{a^2}{b^2}\sin^2\theta}{\cos^2\theta + \frac{a^2}{b^2}\sin^2\theta}} + 1\, d\theta$

$\cancel{= 4a^2\int_0^{\pi/2} \sqrt{\dfrac{(\frac{a^2}{b^2}-1)\sin^2\theta}{1-\sin^2\theta + \frac{a^2}{b^2}\sin^2\theta}} + 1\, d\theta}$

$\cancel{= 4a^2\int_0^{\pi/2} \sin\theta\sqrt{\dfrac{(\frac{a^2}{b^2}-1)}{1-(\frac{a^2}{b^2}+1)\sin^2\theta}} + 1\, d\theta}$



(het gevraagde is +)

$$= -4a^2 \int_0^{\pi/2} \sqrt{\frac{(\frac{a^2}{b^2}-1)\sin^2\theta}{\cos^2\theta + \frac{a^2}{b^2} - \frac{a^2}{b^2}\cos^2\theta}} - 1 \, d\theta$$

$$= -4a^2 \int_0^{\pi/2} \frac{|\sin\theta|\sqrt{\frac{a^2}{b^2}-1}}{\sqrt{-\cos^2\theta(\frac{a^2}{b^2}-1)+\frac{a^2}{b^2}}}\, d\theta + \frac{4a^2\pi}{2}$$

$$= -4a^2 \int_0^{\pi/2} \frac{\sin\theta\sqrt{\frac{a^2}{b^2}-1}}{\sqrt{\frac{a^2}{b^2}\left(1-\left(\frac{\cos\theta\sqrt{\frac{a^2}{b^2}-1}}{a/b}\right)^2\right)}}\, d\theta + 2a^2\pi$$

$$= -\frac{4a^2}{a/b} \int_0^{\pi/2} \frac{\sin\theta\sqrt{\frac{a^2}{b^2}-1}}{\sqrt{1-\left(\frac{\cos\theta\cdot\sqrt{\frac{a^2}{b^2}-1}}{a/b}\right)^2}}\, d\theta + 2a^2\pi$$

Stel $\frac{\cos\theta\sqrt{\frac{a^2}{b^2}-1}}{a/b} = t$

$\Rightarrow \frac{-\sin\theta\sqrt{\frac{a^2}{b^2}-1}}{a/b}\, d\theta = dt$

Als $\theta \to \pi/2$ dan $t \to 0$
Als $\theta \to 0$ dan $t \to \frac{\sqrt{a^2/b^2-1}}{a/b}$

$$= \frac{4a^2}{a/b}\cdot\frac{a}{b} \int_{\frac{\sqrt{a^2/b^2-1}}{a/b}}^{0} \frac{1}{\sqrt{1-t^2}}\, dt + 2a^2\pi$$

$$= 4a^2 \operatorname{Bgsin} 0 - 4a^2 \operatorname{Bgsin}\left(\frac{\sqrt{\frac{a^2}{b^2}-1}}{a/b}\right) + 2a^2\pi$$

$$= -4a^2 \operatorname{Bgsin}\left(\frac{\sqrt{\frac{a^2-b^2}{b^2}}}{a/b}\right) + 2a^2\pi$$

$$= -4a^2 \operatorname{Bgsin}\left(\frac{b}{a}\cdot\frac{1}{b}\sqrt{a^2-b^2}\right) + 2a^2\pi$$

$$= -4a^2 \operatorname{Bgsin}\left(\frac{\sqrt{a^2-b^2}}{a}\right) + 2a^2\pi$$

De totale gevraagde is dit maal 2, dus

$$-8a^2 \operatorname{Bgsin}\left(\frac{\sqrt{a^2-b^2}}{a}\right) + 4a^2\pi.$$

oefening 4.25

H5

parametervoorstelling $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$

stel $\frac{x}{a}=\cos\theta\sin\psi \Leftrightarrow x=a\cos\theta\sin\psi$

analoog voor de andere

$\varphi: K\to\mathbb{R}^3:(\theta,\psi)\mapsto(a\cos\theta\sin\psi,\ b\sin\theta\sin\psi,\ c\cos\psi)$

$\hookrightarrow K=\{(\theta,\psi)\in\mathbb{R}^2 \mid 0\le\theta\le 2\pi,\ 0\le\psi\le\pi\}$

$\frac{\partial\varphi}{\partial\theta}\times\frac{\partial\varphi}{\partial\psi}=(-cb\cos\theta\sin^2\psi, -ac\sin\theta\sin^2\psi, -ab\cos\psi\sin\psi)$

controle oriëntatie: $\left(\frac{\partial\varphi}{\partial\theta}\times\frac{\partial\varphi}{\partial\psi}\right)(\ ,\frac{\pi}{2})$

$=(-cb,0,0)$ in $(a,0,0)$ → verkeerd gericht dus teken wissel!

oppervlakte integraal:

$$\iint_\Sigma \vec{F}\cdot d\vec{\sigma} = \iint_K c\cdot\cos\psi\cdot(ab\cos\psi\sin\psi)\,d\theta\,d\psi$$

$$=\iint_K abc\sin\psi\cos^2\psi\,d\theta\,d\psi$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi}abc\sin\psi\cos^2\psi\,d\psi$$

stel $\cos\psi=t$ als $\psi\to\pi$ dan $\cos\psi\to-1$

$\Rightarrow -\sin\psi\,d\psi=dt$ als $\psi\to0$ dan $\cos\psi\to1$

$$=\int_0^{2\pi}d\theta\int_{-1}^{1}abc\,t^2\,dt=\frac{abc}{3}\int_0^{2\pi}2\,d\theta=\frac{4\pi}{3}abc. \checkmark$$

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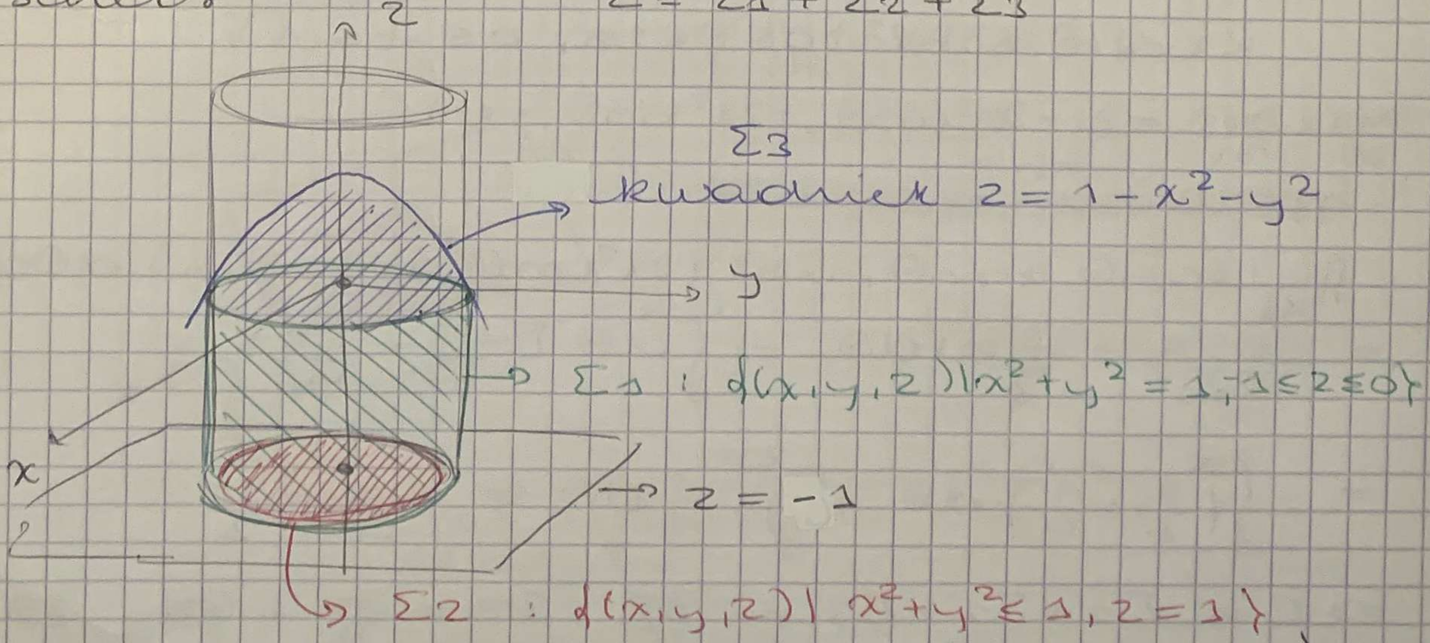
b) schets

$\Sigma=\Sigma_1+\Sigma_2+\Sigma_3$



$\boxed{\Sigma_1}$: parametervoorstelling:

$\varphi_1: K_1 \to \mathbb{R}^3: (\theta, t) \mapsto (\cos\theta, \sin\theta, t)$

$\hookrightarrow K_1$ bepalen $\quad \theta \in [0, 2\pi]$

$t \in [-1, 0]$

$K_1 = \{(\theta, t) \in \mathbb{R}^2 \mid 0 \leq \theta \leq 2\pi; -1 \leq t \leq 0\}$

$$\frac{\partial\varphi_1}{\partial\theta} \times \frac{\partial\varphi_1}{\partial t} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0)$$

goede oriëntatie

- oppervlakte-integraal:

$$\iint_{\Sigma_1} (x, y, z) \, d\vec{\sigma} = \iint_{K_1} \cos^2\theta + \sin^2\theta \, d\theta \, dt$$

$$= \int_{-1}^{0} dt \int_{0}^{2\pi} d\theta = \int_{-1}^{0} 2\pi \, dt = 2\pi$$

$\boxed{\Sigma_2}$: $\varphi_2: K_2 \to \mathbb{R}^3: (\theta, r) \mapsto (r\cos\theta, r\sin\theta, -1)$

$K_2 = \{(\theta, r) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\}$

$$\frac{\partial\varphi_2}{\partial\theta} \times \frac{\partial\varphi_2}{\partial r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = (0, 0, -r)$$

goed georiënteerd

- oppervlakte integraal:

$$\iint_{K_2} +r \, d\theta \, dr = +\int_0^{2\pi} d\theta \int_0^1 r \, dr = +\frac{1}{2} \int_0^{2\pi} d\theta$$

$$= +\frac{2\pi}{2} = +\pi$$

$\boxed{\Sigma_3}$ $\varphi_3: K_3 \to \mathbb{R}^3: (\theta, r) \mapsto (r\cos\theta, r\sin\theta, 1 - r^2)$

$\hookrightarrow K_3$ bepalen: $\quad 1 - r^2 = 0 \Rightarrow r^2 = 1 \Rightarrow r = 1$

$K_3 = \{(\theta, r) \in \mathbb{R}^2 \mid 0 \leq \theta \leq 2\pi, 0 \leq t \leq 1\}$

$$\frac{\partial\varphi_3}{\partial\theta} \times \frac{\partial\varphi_3}{\partial r} = (-2r^2\cos\theta, -2r^2\sin\theta, -r)$$

- oppervlakte-integraal:

$$\iint_{K_3} (r\cos\theta, r\sin\theta, 1 - r^2)(2r^2\cos\theta, 2r^2\sin\theta, r) \, d\theta \, dt$$

$$= \iint_{K_3} r + r^3 \, dr \, d\theta = \int_0^{2\pi} d\theta \left[\frac{r^2}{2} + \frac{r^4}{4}\right]_0^1$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{4}\right) d\theta = \int_0^{2\pi} \frac{3}{4} \, d\theta = \frac{6\pi}{4} = \frac{3\pi}{2}$$

$$\Rightarrow \Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 = \frac{3\pi}{2} + 3\pi = \frac{3\pi}{2} + \frac{6\pi}{2} = \frac{9\pi}{2} \checkmark$$

oefening 4 : c)

HS

parametervoorstelling: het vlak dat deze drie punten bevat wordt gegeven door

$$\alpha \leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \forall r,t \in \mathbb{R}$$

$\varphi : K \to \mathbb{R}^3 : (r,t) \mapsto (1+r+t, -r, -t)$

↳ K bepalen:

$K = \{(r,t) \in \mathbb{R}^2 \mid -1 \leq r \leq 0, -1 \leq t \leq 0\}$

met $\varphi(-1,0) = (0,1,0)$
en $\varphi(0,-1) = (0,0,1)$
en $\varphi(0,0) = (1,0,0)$

$$\frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial t} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = (1,1,1)$$

- oppervlakte-integraal:

$$\iint_\Sigma F \cdot d\sigma = \frac{1}{2}\iint_K (1+r+t, +r^2, -t)(1,1,1)\, dt\, dr$$

$$= \frac{1}{2}\iint_K (1+r+t+r^2-t)\, dt\, dr = \frac{1}{2}\iint_K 1+r+r^2\, dr\, dt$$

$$= \frac{1}{2}\int_{-1}^{0} dt \left[r + \frac{r^2}{2} + \frac{r^3}{3}\right]_{r=-1}^{r=0}$$

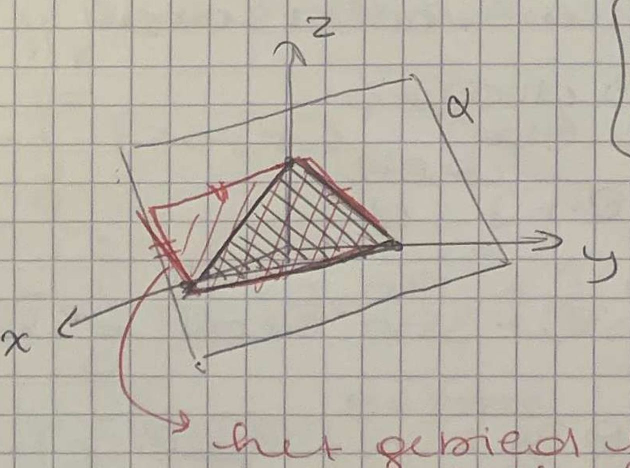
$$= \frac{1}{2}\int_{-1}^{0} -\left(-1+\frac{1}{2}-\frac{1}{3}\right) dt = \frac{1}{2}\int_{-1}^{0} 1 - \frac{1}{2} + \frac{1}{3}\, dt$$

$$= \frac{1}{2}\int_{-1}^{0} \frac{5}{6}\, dt = \frac{5}{12} \checkmark$$

- oppervlakte-int.:

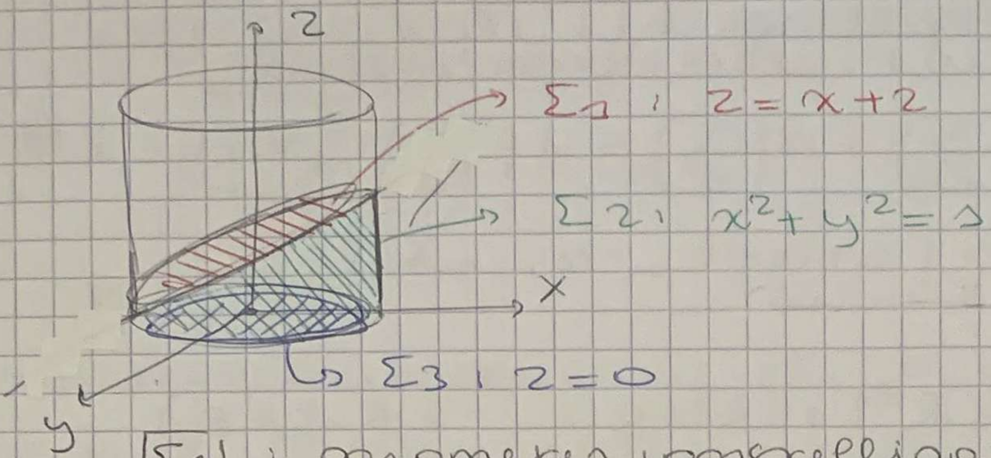
$$\iint_{K_3} 0 \cdot d\theta\, dr = 0$$

totaal: $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$
$= -2\pi$ ✓



het gebied bepaald door K: een rechthoek → dus delen door 2

d) schets



$\boxed{\Sigma_1}$: parametervoorstelling:
$\varphi_1 : K_1 \to \mathbb{R}^3 : (\theta, r) \mapsto (r\cos\theta, r\sin\theta, r\cos\theta + 2)$

K1 bepalen: $\theta \in [0, 2\pi]$, $0 \leq r \leq 1$

$K_1 = \{(\theta, r) \in \mathbb{R}^2 \mid 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 1\}$

$$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -r\sin\theta & r\cos\theta & -r\sin\theta \\ \cos\theta & \sin\theta & \cos\theta \end{vmatrix} = (r, 0, -r) \to \text{goed gericht}$$

- integraal:

$$-\iint_{K_1} (2r\cos\theta, -3r\sin\theta, r\cos\theta + 2)(-r, 0, +r)\, dr\, d\theta$$

$$= -\iint_{K_1} (-2r^2\cos\theta + r^2\cos\theta + 2r)\, d\theta\, dr$$

$$= -\int_0^{2\pi} \left[-\frac{r^3}{3} \cdot \cos\theta + r^3 \right]_{r=0}^{r=1} d\theta$$

$$= -\int_0^{2\pi} \left(-\frac{1}{3}\cos\theta + 1 \right) d\theta = -\left[-\frac{1}{3}\sin\theta + \theta \right]_0^{2\pi}$$

$$= -2\pi$$

$\boxed{\Sigma_2}$: $\varphi_2 : K_2 \to \mathbb{R}^3 : (\theta, t) \mapsto (\cos\theta, \sin\theta, t)$
$K_2 = \{(\theta, t) \in \mathbb{R}^2 \mid 0 \leq t \leq \cos\theta + 2, \ 0 \leq \theta \leq 2\pi\}$

$$\frac{\partial\varphi_2}{\partial\theta} \times \frac{\partial\varphi_2}{\partial t} = (\cos\theta, \sin\theta, 0)$$

$$\iint_{K_2} (2\cos\theta, -3\sin\theta, t)(\cos\theta, \sin\theta, t)\, dt\, d\theta$$

$$= \iint_{K_2} 2\cos^2\theta - 3\sin^2\theta \, dt\, d\theta$$

$$= \int_0^{2\pi} \left[2\cos^2\theta t - 3\sin^2\theta \cdot t \right]_0^{\cos\theta + 2} d\theta$$

$$= \int_0^{2\pi} 2\cos^3\theta + 4\cos^2\theta - 3\sin^2\theta\cos\theta - 6\sin^2\theta \, dt$$

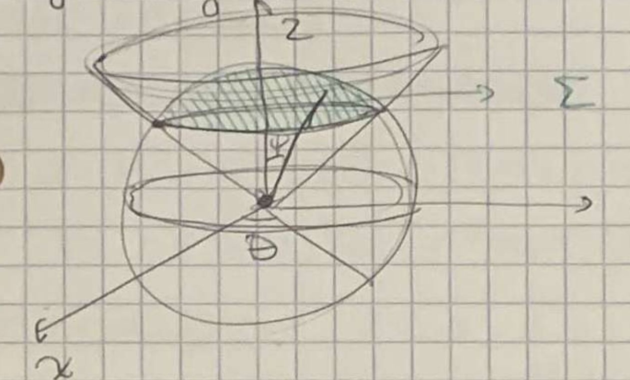
$$= \left[3\sin 2\theta - \sin^3\theta \right]_0^{2\pi} = 0$$

$\boxed{\Sigma_3}$: $\varphi_3 : K_3 \to \mathbb{R}^3 : (\theta, r) \mapsto (r\cos\theta, r\sin\theta, 0)$
$K_3 = \{(\theta, r) \in \mathbb{R}^2 \mid 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 1\}$

$$\frac{\partial\varphi_3}{\partial\theta} \times \frac{\partial\varphi_3}{\partial r} = (0, 0, -r)$$

oefening 4 c)

HS



$\vec{F} = (x, y, 0)$

- parametervoorstelling $x^2+y^2+z^2=1$

$\varphi : K \to \mathbb{R}^3 : (\theta, \psi) \mapsto (\cos\theta \sin\psi, \sin\theta\sin\psi, \cos\psi)$

↳ K bepalen : $z = \sqrt{x^2+y^2}$

$\Rightarrow \cos\psi = \sqrt{\cos^2\theta\sin^2\psi + \sin^2\theta\sin^2\psi}$

$\Rightarrow \cos\psi = \sqrt{\sin^2\psi} = |\sin\psi|$

dus $\psi = \frac{\pi}{4}$, $\psi \in [0, \pi/4]$

$K = \{(\theta, \psi) \in \mathbb{R}^2 \mid 0 \leq \psi \leq \pi/4 , 0 \leq \theta \leq 2\pi\}$

$$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial\psi} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin\theta\sin\psi & \cos\theta\sin\psi & 0 \\ \cos\theta\cos\psi & \sin\theta\cos\psi & -\sin\psi \end{vmatrix}$$

$= (-\cos\theta\sin^2\psi, -\sin\theta\sin^2\psi, -\sin\psi\cos\psi)$

↳ tekenwissel nodig

- oppervlakte-integraal :

$$\iint_\Sigma \vec{F} \cdot d\vec{\sigma} = \iint_K \cos\theta\sin\psi \cdot \cos\theta\sin^2\psi + \sin^2\theta\sin^3\psi \, d\theta\, d\psi$$

$$= \iint_K \cos^2\theta\sin^3\psi + \sin^2\theta\sin^3\psi \, d\theta \, d\psi$$

$$= \iint_K \sin^3\psi \, d\theta\, d\psi = \iint_K (1-\cos^2\psi)\sin\psi \, d\theta\, d\psi$$

$$= \iint_K \sin\psi \, d\theta\, d\psi - \iint_K \cos^2\psi\sin\psi \, d\theta\, d\psi$$

ergens een rekenfout

$$= -\int_0^{2\pi} [\cos\psi]_{\psi=0}^{\psi=\pi/4} d\theta + \frac{1}{3}\int_0^{2\pi} [\cos^3\psi]_{\psi=0}^{\psi=\pi/4} d\theta$$

$$= -\int_0^{2\pi} \frac{\sqrt{2}}{2} - 1 \, d\theta + \frac{1}{3}\int_0^{2\pi} \left(\frac{\sqrt{2}}{2}\right)^3 - 1 \, d\theta$$

$$= -\left[\frac{\sqrt{2}}{2}\theta - \theta\right]_{\theta=0}^{\theta=2\pi} + \frac{1}{3}\left[\frac{2^{3/2}}{8}\theta - \theta\right]_{\theta=0}^{\theta=2\pi}$$

$$= -\pi\sqrt{2} + 2\pi + \frac{1}{3}\,\frac{2^{3/2}}{4}\pi - \frac{2\pi}{3} = -\sqrt{2}\pi + \frac{1}{3}\sqrt{2}\pi - \frac{2\pi}{3} + 2\pi$$

$2^{3/2} = 2\sqrt{2}$

$$= \pi\left(-\frac{2}{3}\sqrt{2} + \frac{4}{3}\right)$$

oefening 4: f) H5

parametervoorstelling $z = 4 - x^2 - y^2$ met $z \geq 0$

$\varphi : K \to \mathbb{R}^3 : (\theta, r) \mapsto (r\cos\theta, r\sin\theta, 4 - r^2)$

$4 - r^2\cos^2\theta - r^2\sin^2\theta = 4 - r^2 = z$

K bepalen: $z \geq 0 \Rightarrow 4 - r^2 \geq 0$
$\Rightarrow -r^2 \geq -4 \Rightarrow 4 \geq r^2$
$\Rightarrow 2 \geq r \geq 0$

$K = \{(\theta, r) \in \mathbb{R}^2 \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$

$$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & -2r \end{vmatrix} = (-2r^2\cos\theta, -2r^2\sin\theta, -r^2)$$

$\left(\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r}\right)(0, 2) = (-8, 0, -4) \to$ tekenwissel

oppervlakte-integraal

$$\iint_\Sigma F \cdot d\sigma = \iint_K (ar\cos\theta, br\sin\theta, c4 - cr^2)(2r^2\cos\theta, 2r^2\sin\theta, r^2)\, d\theta\, dr$$

$$= \iint_K 2ar^3\cos^2\theta + 2br^3\sin^2\theta + 4cr - cr^3\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \left[\frac{2ar^4}{4}\cos^2\theta + \frac{2br^4}{4}\sin^2\theta + 2cr^2 - \frac{cr^4}{4}\right]_0^2$$

$$= \int_0^{2\pi} 8a\cos^2\theta + 8b\sin^2\theta + 8c - 4c\, d\theta$$

$$= \int_0^{2\pi} 8a(1 - \sin^2\theta) + 8b\sin^2\theta + 4c\, d\theta$$

$$= \int_0^{2\pi} 8a - 8a\sin^2\theta + 8b\sin^2\theta + 4c\, d\theta$$

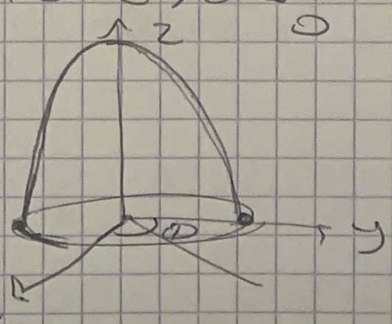
$$= \int_0^{2\pi} (8b - 8a)\sin^2\theta + 4c + 8a\, d\theta \Rightarrow \begin{array}{l} 1 - 2\sin^2\theta = \cos 2\theta \\ \sin^2\theta = \frac{\cos 2\theta - 1}{-2} \end{array}$$

$$= \int_0^{2\pi} (4a - 4b)(\cos 2\theta) - (4a - 4b) + 4c + 8a\, d\theta$$

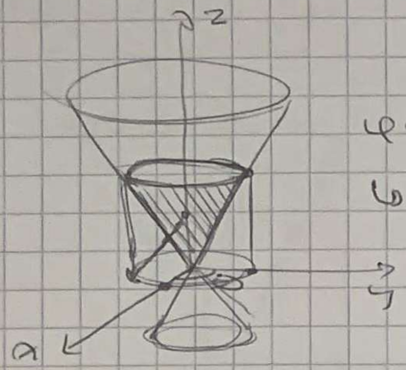
$$= \int_0^{2\pi} (4a - 4b)\cos 2\theta + 4b + 4c + 4a\, d\theta$$

$$= \left[(2a - 2b)\sin 2\theta + 4(a + b + c)\theta\right]_0^{2\pi}$$

$$= 8\pi(a + b + c)$$ ✓



g)



parametervoorstelling kegel:

$\varphi: K \to \mathbb{R}^3 : (\theta, r) \mapsto (r\cos\theta, r\sin\theta, r)$

↳ K bepalen: $\theta \in [0, 2\pi]$

$x^2 + y^2 \le 1$

$\to r^2 \le 1 \iff 0 \le r \le 1$

$K = \{(\theta, r) \in \mathbb{R}^2 \mid 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}$

$$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = (r\cos\theta, r\sin\theta, -r)$$

$$\left(\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r}\right)\left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right) \to \text{OK}$$

oppervlakte-integraal:

$$\iint_\Sigma F \cdot d\sigma = \iint_K (2,5,3) \cdot (r\cos\theta, r\sin\theta, -r)\, d\theta\, dr$$

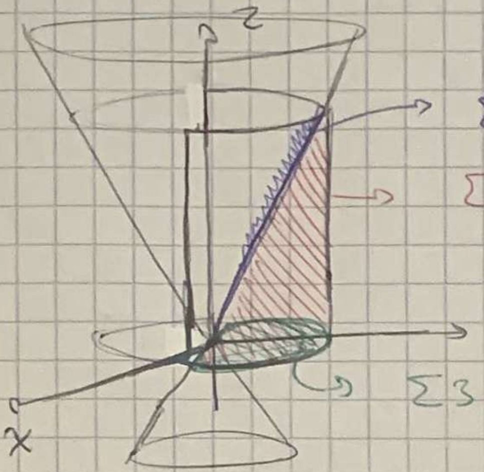
$$= \iint_K 2r\cos\theta + 5r\sin\theta - 3r \, d\theta\, dr$$

$$= \int_0^1 dr \int_0^{2\pi} 2r\cos\theta + 5r\sin\theta - 3r \, d\theta$$

$$= \int_0^1 dr \left[2r\sin\theta - 5r\cos\theta - 3r\theta \right]_0^{2\pi}$$

$$= \int_0^1 (-2\pi 3r)\, dr = \int_0^1 -6\pi r \, dr = \left[-3\pi r^2\right]_0^1 = -3\pi \checkmark$$

oefening 4: h5 H5



$\Sigma_2 : x^2+y^2 = z^2$

$\Sigma_1 : x^2+y^2 = 2x \Leftrightarrow x^2-2x+1-1+y^2 = 0$

$\Leftrightarrow (x-1)^2+y^2 = 1$

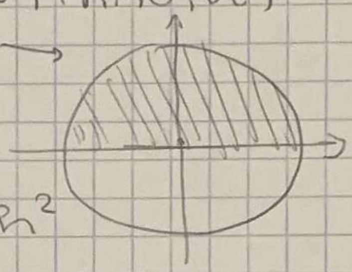
$\Sigma_3 : z = 0$ enkel Σ_1 is gevraagd

$\boxed{\Sigma_1}$ parametervoorstelling:

$\varphi_1 : K_1 \to \mathbb{R}^3 : (\theta, h) \mapsto (\cos\theta + 1, \sin\theta, h)$

K_1 bepalen: $\sin\theta \geq 0$

$\Rightarrow \theta \in [0, \pi]$



$x^2+y^2 = z^2$

$\Rightarrow \cos^2\theta + 2\cos\theta + 1 + \sin^2\theta = h^2$

$\Rightarrow 2\cos\theta + 2 = h^2$

$\Rightarrow \sqrt{2\cos\theta+2} = h$

$K_1 = \{(\theta, h) \in \mathbb{R}^2 \mid 0 \leq \theta \leq \pi,\ 0 \leq h \leq \sqrt{2\cos\theta+2}\}$

$$\frac{\partial\varphi_1}{\partial\theta} \times \frac{\partial\varphi_1}{\partial h} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0)$$ OK.

oppervlakte-integraal:

$$\iint_{\Sigma_1} F\,d\sigma = \iint_{K_1} (\cos\theta+2, \sin\theta-2, h)(\cos\theta, \sin\theta, 0)\,d\theta\,dh$$

$$= \iint_{K_1} \cos^2\theta + 2\cos\theta + \sin^2\theta - 2\sin\theta \,d\theta\,dh$$

$$= \iint_{K_1} 1 + 2(\cos\theta - \sin\theta)\,d\theta\,dh$$

$$= \int_0^\pi d\theta \left[h + 2h(\cos\theta - \sin\theta)\right]_{h=0}^{h=\sqrt{2\cos\theta+2}}$$

$$= \int_0^\pi \sqrt{2\cos\theta+2} + 2\sqrt{2\cos\theta+2}\cos\theta - 2\sin\theta\sqrt{2\cos\theta+2}\,d\theta$$

$$= \int_0^\pi 2\cos\frac{\theta}{2} + 2\left(2\cos\frac{\theta}{2}\right)\cos\theta - 2\sin\theta\sqrt{2\cos\theta+2}\,d\theta$$

$$= \int_0^\pi 2\cos\frac{\theta}{2} + 4\cos\frac{\theta}{2}\left(1 - 2\sin^2\frac{\theta}{2}\right) - 2\sin\theta\sqrt{2\cos\theta+2}\,d\theta$$

$$= \int_0^\pi 6\cos\frac{\theta}{2}\,d\theta + \int_0^\pi -8\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta - 2\int_0^\pi \sin\theta\sqrt{2\cos\theta+2}\,d\theta$$

$$12 - 16\int_0^1 u^2\,du - 2\int_0^{\pi} \sin\theta\sqrt{2\cos\theta+2}\,d\theta$$

stel $2\cos\theta + 2 = t$

$\Rightarrow -2\sin\theta\,d\theta = dt$

als $\theta \to \pi$ dan $2\cos\theta + 2 \to 0$

als $\theta \to 0$ dan $2\cos\theta+2 \to 4$

$$= 12 - \frac{16}{3} + \int_4^0 \sqrt{t}\,dt$$

$$= 12 - \frac{16}{3} + \left[\frac{t^{3/2}}{3/2}\right]_4^0 = 12 - \frac{16}{3} - \frac{4^{\frac{3}{2}}}{3/2}$$

$$= 12 - \frac{16}{3} - \frac{2}{3}\cdot 8 = 12 - \frac{2\cdot 16}{3}$$

$$= \frac{36}{3} - \frac{32}{3} = \frac{4}{3}. \checkmark$$

oefening 1) c) H6: drievoudige

we werken eerst enkel met ons 2 coördi:

$z \in [0, a-x-y]$

$\int_0^{a-x-y} dz = a-x-y$

we beschouwen nu $a-x-y$ als ons scalairenveld voor ons ~~…~~ oppervlakte integraal

 $\to$ parametervoorstelling:

$\varphi : K \to \mathbb{R}^3 : (r,u) \mapsto (r\cos^3 u, r\sin^3 u, 0)$

K bepalen: $K = \{(r,u) \in \mathbb{R}^3 \mid 0 \leq r \leq a, 0 \leq u \leq \pi\}$

$$\frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial u} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos^3 u & \sin^3 u & 0 \\ -3r\cos^2 u \sin u & 3r\sin^2 u\cos u & 0 \end{vmatrix}$$

$= (0, 0, 3r\cos^2 u \sin^4 u + 3r\sin^2 u \cos^4 u)$

$= (0, 0, 3r\cos^2 u \sin^2 u \sin^2 u + 3r\sin^2 u \cos^2 u \cos^2 u)$

$= (0, 0, 3r\cos^2 u \sin^2 u) = (0, 0, \frac{3r\sin^2 2u}{\cancel{\sqrt{2}}\,4})$

$\left\| \frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial u} \right\| = \frac{3r}{\cancel{\sqrt{2}}\,4}\sqrt{\sin^4 2u} = \frac{3r}{\cancel{\sqrt{2}}\,4}\sin^2 2u$

oppervlakte-integraal:

$\iint_\Sigma a-x-y \, dx\,dy = \iint_K (a - r\cos^3 u - r\sin^3 u)\frac{3r\sin^2 2u}{\cancel{\sqrt{2}}\,4} du\,dr$

$= \iint_K \frac{3}{\cancel{\sqrt{2}}\,4}(a\sin^2 2u\, r - r^2\cos^3 u \sin^2 2u - r^2 \sin^3 u \sin^2 2u) du\,dr$

$= \cancel{\ldots}$

$= \int_0^\pi \frac{3}{\cancel{\sqrt{2}}\,4}\left(\frac{a^3}{2}\sin^2 2u - \frac{a^3}{3}\cos^3 u \sin^2 2u - \frac{a^3}{3}\sin^3 u \sin^2 2u\right) du$

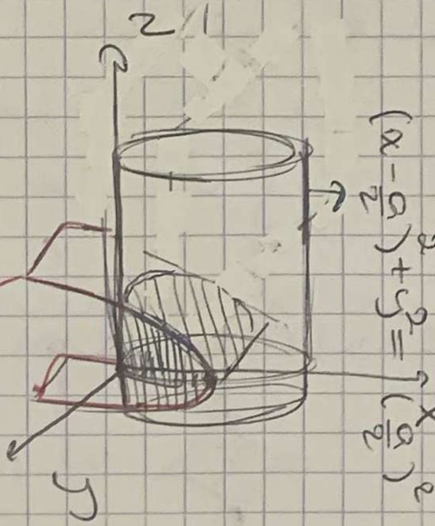
$= \left(\frac{45\pi - 64}{180}\right) a^3 \frac{3}{\cancel{\sqrt{2}}\,4}$

$= \left(\frac{45\pi - 64}{60}\right)\frac{a^3}{\cancel{\sqrt{2}}\,4} = \left(\frac{3\pi}{4} - \frac{16}{15}\right)\frac{a^3}{\cancel{\sqrt{2}}\,4}$

$= \left(\frac{3\pi}{16} - \frac{16}{60}\right) a^3$

d) $x^2 - ax + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$

$\Leftrightarrow \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} = \left(\frac{a}{2}\right)^2$



we beginnen met de z-coördi:

$-\sqrt{b(a-x)} \leq z \leq \sqrt{b(a-x)}$

$z^2 = b(a-x)$

$\int_{-\sqrt{b(a-x)}}^{\sqrt{b(a-x)}} dz = 2\sqrt{b(a-x)}$

↓

scalairenveld voor oppervlakte integraal

we schakelen nu over naar ~~cilinder~~ pool coördi

parametervoorstelling

$\varphi : K \to \mathbb{R}^3 : (r, \theta) \mapsto (r\cos\theta + \frac{a}{2}, r\sin\theta, 0)$

$K = \{(r,\theta) \in \mathbb{R}^2 \mid 0 \leq r \leq \frac{a}{2}, 0 \leq \theta \leq 2\pi\}$

$$\frac{\partial\varphi}{\partial r} \times \frac{\partial\varphi}{\partial\theta} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0, 0, r\sin^2\theta + r\cos^2\theta) = (0,0,r)$$

$\left\| \frac{\partial\varphi}{\partial r} \times \frac{\partial\varphi}{\partial\theta} \right\| = r$

- integraal:

$$\iint_\Sigma 2\sqrt{b(a-x)}\,dx\,dy = \iint_K 2\sqrt{b(\tfrac{a}{2} - r\cos\theta)}\; r\,dr\,d\theta$$

$$= 2\int_0^{2\pi} d\theta \int_0^{a/2} \sqrt{b(\tfrac{a}{2} - r\cos\theta)}\; r\,dr$$

$= \dots$ ↳ te moeilijk

schakel enkel om naar parameters als ze er expliciet vragen

H6

$$y = \pm\sqrt{\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2}$$

$$= \pm\sqrt{\frac{a}{4} - \left(x^2 - ax + \frac{a}{4}\right)} = \pm\sqrt{-x^2 + ax}$$

$$= \pm\sqrt{x(a-x)}$$

$$-\frac{a}{2} \leq x \leq \frac{a}{2}$$

integraal:

$$\int_{-a/2}^{a/2} dx \int_{-\sqrt{x(a-x)}}^{\cancel{\sqrt{x(a-x)}}\ 0} 2\sqrt{b(a-x)}\, dy$$

$$= \int_{-a/2}^{a/2} 2\sqrt{b(a-x)} \cdot \sqrt{x(a-x)}\, dx$$

$$= 2\int_{-a/2}^{a/2} \sqrt{bx(a-x)^2}\, dx$$

$$= 2\int_{-a/2}^{a/2} \sqrt{b}\sqrt{x} \cdot (a-x)\, dx$$

$$= 2\int_{-a/2}^{a/2} \sqrt{b}\sqrt{x}\, a - x\sqrt{x}\sqrt{b}\, dx$$

$$= 2\int_{-a/2}^{a/2} \sqrt{b}\sqrt{x}\, a - x^{\frac{3}{2}}\sqrt{b}\, dx$$

$$= 2\sqrt{b}\left[\frac{x^{3/2}}{3/2}a - \frac{x^{5/2}}{5/2}\right]_{x=0}^{x=\frac{a}{2}}$$

$$= 4\sqrt{b}\left(\frac{a^{3/2}a}{3/2} - \frac{2a^{5/3}}{5}\right)$$

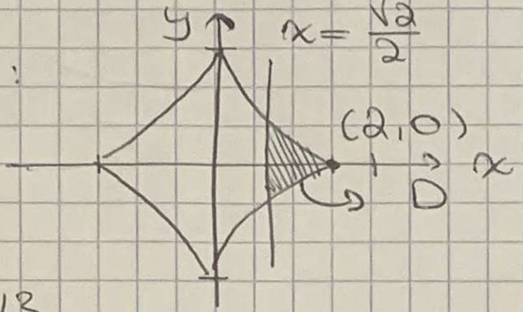
$$= 4\sqrt{b}\left(\frac{2a^{5/3}}{3} - \frac{2a^{5/3}}{5}\right)$$

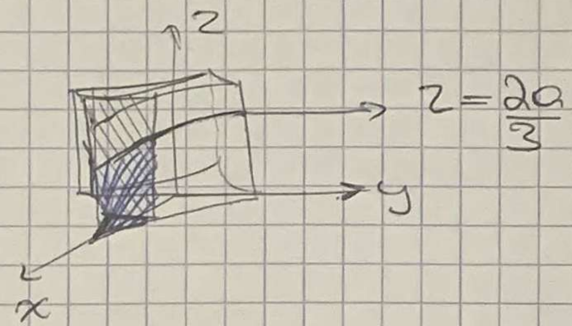
$$= 4\sqrt{b}\left(\frac{10a^{5/3}}{15} - \frac{6a^{5/3}}{15}\right)$$

$$= \frac{16}{15}\sqrt{b}\, a^{5/3}$$

oefening 1) d)   HG

D:





$$\int_0^{2a/3} dz = \frac{2a}{3}$$ → om scalairveld voor om oppervlakte integraal

parametervoorstelling gebied D

$\varphi : K \to \mathbb{R}^3 : (\theta, r) \mapsto (r\cos^3\theta,\ r\sin^3\theta,\ 0)$

K bepalen: $\frac{\sqrt{2}}{2} \le r \le 2$, $0 \le \theta \le \frac{\pi}{4}$

$$\frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -3r\cos^2\theta\sin\theta & 3r\sin^2\theta\cos\theta & 0 \\ \cos^3\theta & \sin^3\theta & 0 \end{vmatrix}$$

$= (0, 0, -3r\sin^2\theta\cos^2\theta\cos^2\theta - 3r\cos^2\theta\sin^2\theta\sin^2\theta)$
$= (0, 0, -3r\sin^2\theta\cos^2\theta)$
$= (0, 0, -\frac{3r}{4}\sin^2 2\theta)$

$\left\| \frac{\partial\varphi}{\partial\theta} \times \frac{\partial\varphi}{\partial r} \right\| = \frac{3r}{4}\sin^2 2\theta$

integraal:

$$\iint_D \frac{2a}{3}\,dx\,dy = 2\iint_K \frac{2a}{3}\cdot\frac{3r}{4}\sin^2 2\theta\, d\theta\, dr$$

$$= 2\int_0^{\pi/4} d\theta \left[\frac{2a}{2}\frac{r^2}{4}\sin^2 2\theta\right]_{r=\frac{\sqrt{2}}{2}}^{r=2}$$

$$= \frac{2a}{4}\int_0^{\pi/4} 4\sin^2 2\theta - \frac{1}{2}\sin^2 2\theta\, d\theta$$

$$= \frac{2a}{4}\int_0^{\pi/4} \frac{7}{2}\sin^2 2\theta\, d\theta$$

$$= -\frac{2a}{4}\int_0^{\pi/4} \frac{7}{2}\left(\frac{\cos 4\theta - 1}{2}\right) d\theta$$

$$= -\frac{7}{2}\frac{a}{4}\left[\frac{\sin 4\theta}{4} - \frac{1}{2}\right]_0^{\cancel{\pi/4}\ \pi/2}$$

$$= -\frac{7}{2}\frac{a}{4}\left(-\frac{\pi}{8}\right) = \frac{7a\pi}{16}$$

$= \cancel{-\frac{7}{2}\frac{a}{4}(}$

deze oefeningen wat gefumbeld... succes ermee xx oefening 1.

H7: Extrema

a) kritieke punten:

$(\nabla f)(x,y) = \vec{0}$

$\rightarrow (2x, 0) = \vec{0}$ dus $2x = 0 \Rightarrow x = 0$
y maakt niet uit

dus het punt $(0, y)$ is een kritiek punt

Hessiaans determinant:

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(0,y) & \frac{\partial^2 f}{\partial x \partial y}(0,y) \\ \frac{\partial^2 f}{\partial x \partial y}(0,y) & \frac{\partial^2 f}{\partial y^2}(0,y) \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$\rightarrow$ inconclusive

maar intuïtief weten we dat $(0,y)$ een lokaal minimum is.

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c)

$(\nabla f)(x,y) = (2x, 2y-2)$

$\rightarrow \begin{cases} 2x = 0 \\ 2y - 2 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases}$ $(0,1)$: kritiek punt

Hessiaans determinant:

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(0,1) & \frac{\partial^2 f}{\partial x \partial y}(0,1) \\ \frac{\partial^2 f}{\partial x \partial y}(0,1) & \frac{\partial^2 f}{\partial y^2}(0,1) \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

en $\frac{\partial^2 f}{\partial x^2} = 2 > 0$ dus $f(0,1)$ is een lokaal minimum

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d)

$(\nabla f)(x,y) = (2x, -2y)$

↳ het punt $(0,0)$ is een kritiek punt

Hessiaans determinant:

$$\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$$ dus het punt $(0,0)$ is een zadelpunt.

e) $f(x,y) = x^2 - xy + y^2$

$(\nabla f)(x,y) = (2x - y, 2y - x)$

$$\begin{cases} 2x - y = 0 \\ 2y - x = 0 \end{cases} \qquad \left(\begin{array}{cc|c} +1 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 3 & 0 \end{array}\right)$$

$$\rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \sim \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

het punt $(0,0)$ is een kritiek punt

Hessiaanse determinant

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$$

en $\frac{\partial^2 f}{\partial x^2} > 0$ dus het punt $(0,0)$ is een lokaal minimum

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f)

$f = e^{x - y^2 - x^3/3}$

$(\nabla f)(x,y) = \left(e^{x - y^2 - x^3/3} \cdot (\overset{1}{\cancel{x}} - x^2), e^{x - y^2 - x^3/3}(-2y)\right)$

omdat exponentiële functies nooit nul worden gaan we het volgende stelsel na:

$$\begin{cases} \overset{1}{\cancel{x}} - x^2 = 0 \\ -2y = 0 \end{cases} \rightarrow \begin{cases} x = -1 \vee 1 - x = 0 \Leftrightarrow x = 1 \\ y = 0 \end{cases}$$

dus we hebben $(-1,0)$ en $(1,0)$ als kritieke punten